Mathematical Reviews

Vol. 21, No. 9 **OCTOBER, 1960** **Reviews 5529–6310**

TABLE OF CONTENTS

AUTHOR INDEX

(Continued on cover 3)

Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4 (82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. In references to reviews, the sign # refers to the number of the review.

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179
815
529
737
648
154
075
536
819
684
771
778
947
838
590
148
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Mathematical Reviews

Vol. 21, No. 9 — October, 1960 — Reviews 5529–6310

HISTORY AND BIOGRAPHY

See also 5546, 5968.

5529:

★**Clagett, Marshall. (Editor) Critical problems in the history of science.** Proceedings of the Institute for the History of Science at the University of Wisconsin, September 1–11, 1957. The University of Wisconsin Press, Madison, Wis., 1959. xiv+555 pp. $5.00.

Sixteen major papers and 19 commentaries by American, English, and Dutch historians of science. None of the papers are concerned directly with the history of mathematics, and for the most part mathematics gets only incidental attention. The exceptions are a paper by Joseph T. Clark urging that problems in the history of science be analyzed in terms of modern methods and concepts, especially those of mathematics (e.g., the concept of isomorphism), and a paper by Derek J. de S. Price in which the author makes the startling claim that Copernican and Ptolemaic theories were "formally equivalent" (page 198). Current controversy over the role of logical and set-theoretic notions in the teaching of elementary mathematics is reflected in opinions by Clark (page 110) and I. E. Drabkin (page 151). Generally the mathematical content of the book is poorly indexed. The book testifies to the neglect of mathematics by historians and of history by mathematicians. *K. O. May* (Northfield, Minn.)

5530:

Gerardy, Theo. Der Briefwechsel zwischen Carl Friedrich Gauss und Carl Ludwig von Lecoq. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. II **1959**, 37–63.

5531:

Gerardy, Theo. Ein unveröffentlichter Brief von Carl Friedrich Gauss an Alexander von Humboldt. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. II **1959**, 64–66.

5532:

★**Turing, Sara. Alan M. Turing.** W. Heffer & Sons, Ltd., Cambridge, 1959. xiv+157 pp. (7 plates) 21s.

This book, written by the mother of the late English logician and mathematician A. M. Turing (1912–1954), is principally concerned with a chronologically arranged biography of him. Unfortunately it was necessary for Mrs. Turing to omit all but the superficial mention of her son's activities in the Foreign Office during the Second World War (six years!). The reviewer, who has read much of A. M. Turing's abstract works, is enthusiastically looking forward to the day when Turing's war-work is finally made available. The book has a brief section dealing cursorily with some of A. M. Turing's technical work on computing machines and morphogenesis.

In particular, chapter 1 (7 pp.) deals with Turing's family background. Chapters 2 and 3 (31 pp.) consider his pre-university days in Europe. Chapter 4 (10 pp.) places in evidence some of his undergraduate activities while he was at Cambridge University. Chapter 5 (5 pp.) is a brief sketch of his life at Princeton University, from which he obtained a Ph.D. in mathematics (1938). The rest of part I of the book (66 pp.) deals with some of A. M. Turing's characteristics, his professional life and his untimely death. Part 2 of the book (26 pp.) is technical in nature and is concerned with computing machinery, morphogenesis and a bibliography.

The book has six full-page photographic plates of A. M. Turing at various ages and a plate of a specimen of his handwriting.

The reviewer especially recommends the book to those who wish to obtain a fuller understanding of Turing's logical and mathematical investigations. *A. A. Mullin* (Urbana, Ill.)

5533:

Borůvka, Otakar; Čermák, Jiří; Radochová, Věra; und Frank, Ludvík. Mathias Lerch und sein Werk auf dem Gebiete der mathematischen Analysis. Práce Brn. Československ. Akad. Věd **29** (1957), 417–540. (Czech. German summary)

This detailed survey (constituting the entire issue 10–11 of the journal) consists of six articles by the individual authors, on Lerch's work in (1) general function theory, (2) infinite series, (3) the Gamma function, (4) elliptic functions, (5) integral calculus, and (6) his controversy with Pringsheim.

5534:

Balázs, János. The scientific work of the late Michael Fekete. Mat. Lapok **9** (1958), 197–224. (Hungarian)

5535:

Dávid, Lajos. In memoriam Wolfgang Bolyai. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **9** (1959), 215–236. (Hungarian)

5536:

Collingwood, E. F. Émile Borel. J. London Math. Soc. **34** (1959), 488–512.

This survey of the life [1871–1956] and work [317 publications] of a distinguished figure in modern mathematics

includes an up-to-date extension of the bibliography in the Borel jubilee volume *Selecta* [Gauthiers-Villars, Paris, 1940; MR 1, 128].

5537:

Borkowski, L.; and Słupecki, J. The logical works of J. Łukasiewicz. Studia Logica 8 (1958), 7–56.

This valuable survey of Łukasiewicz's work is divided into three parts: Methodology of empirical sciences (especially induction and probability), mathematical logic, significance of logic and its relation to other sciences. Łukasiewicz's work in the history of logic is not dealt with in this paper. *H. Freudenthal* (Utrecht)

5538:

Kotarbiński, Tadeusz. Jan Łukasiewicz's works on the history of logic. Studia Logica 8 (1958), 57–62.

Łukasiewicz's research on Aristotelian and Stoic logic and its influence on research of other scholars.
H. Freudenthal (Utrecht)

5539:

★Pompeiu, Dimitrie. Opera matematică. Editura Academiei Republicii Populare Romîne, Bucharest, 1959. xxxi + 533 pp. (1 plate) Lei 30,20.

Includes a preface by S. Stoïlow, a brief review of Pompeiu's life and work by S. Marcus, and almost 150 articles (almost all in French).

5540:

Davis, Philip J. Leonhard Euler's integral: A historical profile of the gamma function. Amer. Math. Monthly **66** (1959), 849–869.

An historical introduction to the gamma function.
H. Freudenthal (Utrecht)

5541:

★Eulerus, Leonhardus. Opera omnia. Series secunda. Opera mechanica et astronomica. Vol. VII. Commentationes mechanicae ad theoriam motus punctorum pertinentes. Volumen posterius. Edidit C. Blanc. Societas Scientiarum Naturalium Helveticae, Lausanne, 1958. viii + 327 pp.

Most of the memoirs in this volume were written in the last decade of Euler's life and concern special mechanical problems of no great interest. At the end appear the papers E826–E829, first published in 1862; these are preliminary studies toward the great essay E86 on relative motion, published in 1746 and described in the review of the preceding volume [vol. VI, 1957; MR **20** #6970].
C. Truesdell (Bloomington, Ind.)

5542:

★Ляпунов, А. М. Академик А. М. Ляпунов; собрание сочинений. Том IV. [Lyapunov, A. M. Academician A. M. Lyapunov; collected works. Vol. IV.] Izdat. Akad. Nauk SSSR, Moscow, 1959. 645 pp. (2 plates) 29.60 rubles.

This volume is devoted to Lyapunov's work on the determination of equilibrium figures for rotating homogeneous fluids. It contains his four major memoirs on the subject, which appeared originally in French between 1906 and 1913. *H. A. Antosiewicz* (Los Angeles, Calif.)

GENERAL

5543:

★Andree, Richard V. Selections from modern abstract algebra. Henry Holt and Co., New York, 1958. xii + 212 pp. $6.50.

This is an elementary textbook that provides a brief introduction to some of the simpler concepts of abstract algebra. It is addressed to students at the sophomore level. The chapter headings are: 1. Number theory and proof; 2. Equivalence and congruence; 3. Boolean algebra; 4. Groups; 5. Matrices; 6. Linear systems; 7. Determinants; 8. Groups, rings and fields; 9. More matrix theory.

The book is designed to titillate the student's mathematical appetite rather than to satisfy it. For the able student there are problems that treat theoretical results not covered in the text and frequent suggestions for further reading. *D. C. Murdoch* (Vancouver, B.C.)

5544:

★Picone, Mauro; e Fichera, Gaetano. Trattato di analisi matematica. Vols. I, II. Tumminelli Editore, Roma, 1954, 1955. Vol. I, vi + 520 pp. 6,500 Lire; Vol. II, 883 pp. 10,000 Lire.

Questo trattato sviluppa in forma assai ampia le idee e i metodi che il Picone ha costantemente seguiti nell'insegnamento dell'Analisi Matematica, a partire delle sue "Lezioni di Analisi infinitesimale" [Circ. Mat. Catania **3** (1923), 1–31]. A fondamento dell'opera è posta la "teoria dei limiti" del Picone, estremamente generale e divenuta oggi della più moderna attualità.

La grande generalità nell'impostazione dei concetti, insieme alla costante preoccupazione di rendere "utili" le dimostrazioni ai fini del calcolo numerico, costituisce una notevole caratteristica di questa opera, in armonia con l'esperienza degli Autori, nell'Istituto per le Applicazioni del Calcolo, fondato dal Picone nel 1927 e da lui diretto da quell'anno.

Passiamo ora in rassegna, rapidamente, il contenuto del primo volume, il quale consta di cinque capitoli.

Il Cap. I (matrici e determinanti) tratta gli argomenti: numeri complessi, analisi combinatoria, vettori matrici e determinanti, calcolo algebrico con matrici, forme quadratiche ed hermitiane, risoluzione numerica dei sistemi lineari. La trattazione risulta molto completa; vi si trova, ad esempio, il teorema di Hadamard sul massimo modulo di un determinante.

Il Cap. II (limiti per una variabile, insiemi di punti, funzioni) contiene la teoria degli insiemi ordinati di operazioni e la teoria dei limiti per una variabile ordinata, cui si è già accennato: tratta inoltre, nello spirito di tale teoria, le nozioni ed operazioni fondamentali negli insiemi di punti e sulle funzioni di variabili reali.

Il Cap. III (derivate e differenziali per le funzioni reali di punto) è dedicato al calcolo differenziale per le funzioni di una o più variabili, pervenendo ai teoremi delle funzioni implicite, sulla dipendenza e indipendenza funzionale, e sull'inversione.

Il Cap. IV (generalità sull'integrazione) svolge i fondamenti della teoria dell'estensione degli insiemi di punti e dell'integrazione secondo Riemann per le funzioni di più

variabili: più dettagliamente la teoria e i metodi per le funzioni di una variabile.

Il Cap. V (funzioni olomorfe di variabili complesse) contiene una introduzione alla teoria delle funzioni analitiche, con le definizioni e le proprietà delle funzioni elementari nel campo complesso.

Il vol. II del trattato consta di cinque capitoli. Il Cap. I (divisibilità fra polinomi) è dedicato alla teoria della divisibilità fra polinomi in una o più variabili, al teorema fondamentale dell'algebra e alla teoria dell'eliminazione.

Il Cap. II (calcolo delle radici di un'equazione algebrica a un'incognita) è particolarmente notevole. Oltre al teorema di Budan-Fourier e alla regola dei segni di Cartesio vi si trovano infatti i teoremi di Routh e di Hurwitz sull'ubicazione delle radici complesse. A questi risultati, dati in forma più ampia di quella consueta, segue una ricca ed originale esposizione dei metodi di separazione e di calcolo numerico delle radici delle equazioni algebriche: in particolare il fondamentale metodo di Graeffe.

Il Cap. III (complementi alla teoria dell'integrazione delle funzioni) tratta estesamente la teoria dell'integrazione (anche per funzioni non continue) sulla retta, sul piano e sullo spazio. Da rilevare la trattazione esauriente dell'integrazione curvilinea, delle forme differenziali lineari, dei teoremi di Cauchy e Morera, l'analisi accurata delle condizioni di validità della formula di Green (deducendone, tra l'altro, il teorema di riduzione) e del teorema di Stokes. Interessante pure la teoria dell'area di una superficie, e del cambiamento di variabili degli integrali multipli, svolta in modo esauriente e rigoroso.

Il Cap. IV (le serie) contiene classici criteri di convergenza sulle serie numeriche, semplici o multiple, un'ampia esposizione sulle serie di potenze (con nuove formule integrali del resto), il metodo di Goursat per la risoluzione dei sistemi non lineari con approssimazioni successive, un procedimento per il calcolo numerico dei massimi e minimi delle funzioni di variabili reali. Sono esposti inoltre dei complementi sulle funzioni analitiche, e nozioni ed esempi fondamentali sui prodotti infiniti, cui si riallacciano le funzioni euleriane.

Il Cap. V (equazioni differenziali) contiene nei primi due paragrafi i teoremi di esistenza ed unicità per i sistemi ordinari, anche nel campo complesso, ed una interessante e suggestiva esposizione di tipici problemi sulle equazioni a derivate parziali della fisicamatematica (argomento questo coltivato con particolare successo da entrambi gli Autori): vi si trovano, tra l'altro, i lemmi di Green e di Stokes, il teorema di Gauss e le applicazioni all'equazione di Laplace. Chiude il capitolo una approfondita trattazione dei sistemi lineari, in particolare a coefficienti costanti (anche di tipo non normale). A conclusione della lettura di questo trattato, può ben dirsi che esso costituisce un'opera di largo respiro, condotta sulla base del massimo rigore, originale nell'impostazione fondamentale ed in molti particolari.

L. Amerio (Milan)

5545:

★**Hyman, Charles. German-English mathematics dictionary.** Interlanguage Dictionaries Publishing Corp., New York, 1960. 131 pp. $8.00.

From the foreword: "The terms that constitute this dictionary have been collected from textbooks and other authoritative works on the basis of long years of experience in scientific and technical translation ... The dictionary contains more than 8500 entries."

5546:

★**Bachet, Claude-Gaspar. Problèmes plaisants & délectables, qui se font par les nombres.** 5ième éd. Revue, simplifiée et augmentée par A. Labosne. Librairie Scientifique et Technique Albert Blanchard, Paris, 1959. viii + 243 pp. Paperbound: 9 NF.

Preface by J. Itard. For a critique of the author (1581–1638), see Collet et Itard, Rev. Hist. Sci. Appl. **1** (1947), 26–50 [MR **10**, 420].

5547:

★**Mordell, L. J. Reflections of a mathematician.** Canadian Mathematical Congress, Montreal, Que., 1959. viii + 50 pp. (1 plate)

Addressed to a general audience, this monograph sets forth, in an informal style, many of the day-to-day concerns and working habits of the professional mathematician.

5548:

★**Hadamard, Jacques. Essai sur la psychologie de l'invention dans le domaine mathématique.** Traduit de l'anglais par Jacqueline Hadamard. Première édition française revue et augmentée par l'Auteur. Librairie Scientifique Albert Blanchard, Paris, 1959. 135 pp. Paperbound: 8 NF.

Two brief appendices have been added to the American editions [Princeton Univ. Press, 1949; Dover, N.Y., 1954; MR **10**, 423; **16**, 3].

5549:

Pogorzelski, H. A. Generalization of a Peano symbol. Amer. Math. Monthly **66** (1959), 885.

The author points out symbols, $\upharpoonright$, $\downharpoonright$ and $\uparrow$ or $\downharpoonleft$, that enable one to print everything on the same line provided the page is wide enough. One of these symbols had been introduced by Peano [*Formulaire de mathématique*, vol. 2, Bocca, Turin 1898; MR −**41**, 84]. {The symbols are presented for the consideration of mathematicians in easing the printer's problems such as, e.g., stated by T. W. Chaundy, P. R. Barrett and C. Batey [*The Printing of Mathematics*, Oxford Univ. Press, London, 1954; MR **15**, 1011].}

P. Samuel (Clermont-Ferrand.)

LOGIC AND FOUNDATIONS

See also 5537, 5538, 5647.

5550:

Kreisel, Georg. Hilbert's programme. Dialectica **12** (1958), 346–372. (German summary)

The paper contains an interesting discussion concerning Hilbert's program. The author emphasizes that the consistency problem is associated by Hilbert [D. Hilbert, *Grundlagen der Geometrie*, Springer, Leipzig-Berlin, 1930] with understanding the concept of infinity, which requires

the elimination of non-finitist procedure from proofs of finitist statements, i.e., the reduction to finitist methods. The failure of Hilbert's plan leads to a hierarchy of Hilbert programs consisting in the reduction to progressively less elementary, but still constructive, methods instead of to finitist ones. The mathematical proof of the failure of Hilbert's original plan requires a definition of finitist proofs. For certain parts of non-constructive mathematics, the Hilbert program has been carried out in the sense intended by Hilbert. For this aim a syntactic analysis has been applied. The author describes three methods of syntactic analysis (the ε-substitution method, the method of Gentzen and Herbrand, Gödel's method), their mathematical significance (including applications to the independence problem), and their connection with the modified Hilbert program. Remarks concerning finitist proofs and the completeness of the first-order predicate calculus conclude the paper. Successive approaches and interesting unsolved problems are indicated. *H. Rasiowa* (Chicago, Ill.)

5551:

Specker, Ernst. Dualität. Dialectica **12** (1958), 451–465. (French and English summaries)

Consider elementary (first-order) theories built up in the style of plane projective geometry. For any sentence S of such a theory, let S^* denote the dual of S (e.g., the result of interchanging the predicates for point and line). For any model $\mathfrak{M}$ of the theory, let $\mathfrak{M}^*$ be the dual model. The author very rightly distinguishes three forms of the principle of duality for axiom systems for this type of theory: (I) if a sentence S is provable from the axioms, then so is S^*; (II) for any sentence S, the equivalence $S \equiv S^*$ is provable from the axioms; (III) every model $\mathfrak{M}$ satisfying the axioms is isomorphic to its dual $\mathfrak{M}^*$. It is obvious that for any given axiom system (III) implies (II) implies (I). It is known for the ordinary axioms of incidence that (I) is valid but is not (II) (references are given in the paper). Of course, if Pappus' Theorem is added as a new axiom, then (III) is valid; however, Desargues' Theorem alone is not sufficient, except for finite models. The author constructs a simpler axiom system for which the proof that (I) is valid but (II) is not, is easier than for projective geometry. In the case of complete axiom systems (I) and (II) are equivalent. The example given also yields a system satisfying (I) but for which no model is ever isomorphic to its dual. Another example shows that (II) can be valid without (III); however, in an added remark at the end of the paper the author states without proof the highly interesting result that every system for which (II) is valid must possess at least one model isomorphic to its dual. {R. L. Vaught and the reviewer have noticed that this fact may be deduced as a direct consequence of a theorem of A. Robinson [Indag. Math. **18** (1956), 47–58; MR **17**, 1172].} The author further points out the amusing fact that if each of the sentences of the form $S \equiv S^*$ is separately consistent with an axiom system, then all these sentences may be simultaneously adjoined as new axioms preserving consistency.

These ideas of duality can also be fruitfully applied to a more general situation than that resulting from a simple interchange of two predicates. The author gives details for the case of the simple theory of (positive and negative) types. The axioms used are Extensionality and the Comprehension Schema at every type level. If S is a sentence of this theory, then S^* is the result of increasing the type level of all variables in S by one unit. If it would be consistent to add all the equivalences of the form $S \equiv S^*$ to the above-mentioned axioms, then by the author's announced result there would be a model having a membership preserving 'automorphism' shifting the type levels by one unit. This model of type theory would be rather remarkable, for, as the author demonstrates, it would easily yield a model for Quine's system of set theory known as New Foundations (and conversely). Hence, the consistency problem for New Foundations has been reduced to a seemingly simpler and, to the reviewer's mind at least, more understandable problem. It remains to be seen, however, whether anyone can find a model for the author's theory of types. *Dana Scott* (Chicago, Ill.)

5552:

★**Engeler, Erwin. Untersuchungen zur Modelltheorie.** Promotionsarbeit. Art. Institut Orell Füssli AG, Zürich, 1958. 28 pp.

The notion of a model is meant in the non-formal sense as a relational system satisfying given first-order axioms. [See Tarski, Indag. Math. **16** (1954), 572–581, 582–588; MR **16**, 554]. Nevertheless, in the last § 5, an outline is given of a formal definition of the notion of satisfaction in the axiomatic set theory of Bernays. (Instead of Gödel-numbers certain n-tuples of finite sets are used to represent formulae, but no further use is made of them.)

Maximality and minimality (in the sense of inclusion) of models is dealt with in § 2 and § 3. It is shown that every (noncontradictory) first-order axiomatic system can be enlarged so as to have a minimal model (by adding primitive operations corresponding to existential axioms, in the well-known sense of Skolem). The known theorem of extending infinite models is proved by a direct transfinite induction, not using the Gödel-Malcev completeness theorem explicitly. Less known than these results may be the notion of satisfaction of certain countably infinite disjunctions, introduced by the author in § 4. Unfortunately, no exact formation-, transformation- and semantical rules are given, so that the reviewer sees some ambiguities, e.g., for universal quantifiers as applied to these infinite disjunctions. *L. Rieger* (Prague)

5553:

Mal'cev, A. I. On small models. Dokl. Akad. Nauk SSSR **127** (1959), 258–261. (Russian)

For a model $\mathfrak{M} = \langle M, \{P_i\}, \{a_j\} \rangle$, where the P_i are predicates and the a_j individual constants, the cardinality of the P_i and a_j is called the order of $\mathfrak{M}$. A model is called regular if it is infinite and its power is greater than or equal to its order. Non-regular models are said to be small. It is known that any regular model $\mathfrak{M}$ in a class K of models of an axiomatic theory can be imbedded in another model in K of arbitrary power $\geq$ the power of $\mathfrak{M}$. The author proves the following partial generalization of this result to small models. Let K be a class of models of an axiomatic theory. Theorem 1: If K contains an infinite model $\mathfrak{M}$ of power $\mathfrak{m}$, then $\mathfrak{M}$ has a proper K-extension of power $\mathfrak{m}^{\aleph_0}$; and if there are K-models of powers $\mathfrak{m}_1 < \mathfrak{m}_2 < \cdots$, then there is a K-model of power $\mathfrak{n}$ such that $\mathfrak{m}_1 + \mathfrak{m}_2 + \cdots \leq \mathfrak{n} \leq \mathfrak{m}_1 \cdot \mathfrak{m}_2 \cdots$. From the generalized continuum hypothesis (G.C.H.), it then follows

that every infinite K-model $\mathfrak{M}$ has a K-extension of any power greater than the power of $\mathfrak{M}$. If K is a class of models, let K_α, K^β, $K_\alpha{}^\beta$ be the class of K-models of power $\mathfrak{m}$ satisfying $\alpha \leq \mathfrak{m}$, $\mathfrak{m} \leq \beta$, $\alpha \leq \mathfrak{m} \leq \beta$, respectively. Theorem 2: If K and L are classes of models of axiomatic theories of the same type, and there is some infinite cardinal α for which $K_\alpha{}^\alpha \subseteq L$, then $K_{\aleph_0} \subseteq L$ if $\alpha \geq$ order of K, and $K_{\aleph_0}{}^\alpha \subseteq L$ otherwise. (The second part of this assertion assumes the G.C.H.) The proof of theorem 1 reduces the problem to the case where all axioms are universal and uses the second ε-theorem. (The following examples show that, for finite and countable models, theorem 1 contains the "best possible result". There is a class K_1 containing finite models of arbitrarily large finite power, but containing no denumerable model. There is a class K_2 containing a denumerable model $\mathfrak{U}$ such that any proper K_2-extension of $\mathfrak{U}$ has at least the power of the continuum.) *E. Mendelson* (New York, N.Y.)

5554:

Taimanov, A. D. Class of models closed with respect to direct union. Dokl. Akad. Nauk SSSR **127** (1959), 1173–1175. (Russian)

The conjecture of A. Horn [J. Symbolic Logic **16**, 14–21 (1951); MR **12**, 662] is disproved. A necessary and sufficient condition that an axiom be reducible to one of the Horn type is presented. This criterion permits the citation of an example refuting the conjecture in general and also yields as a corollary the result of Bing [Proc. Amer. Math. Soc. **6** (1955), 836–846; MR **17**, 226] which established special cases of the conjecture. Several other theorems give sufficient conditions that a system of axioms be multiplicatively closed. *R. A. Good* (College Park, Md.)

5555:

Lyndon, Roger C. An interpolation theorem in the predicate calculus. Pacific J. Math. **9** (1959), 129–142.

We consider sentences of the predicate calculus (with or without equality) in which as sentential connectives only the signs for negation, conjunction, and disjunction, together with the signs 0 and 1 for truth and falsehood, are admitted. For each occurrence of a relation symbol in a sentence S, there is a unique maximal chain of well formed formulas, all containing the given occurrence and each occurring as a proper part of the next; the given occurrence of the relation symbol will be called positive if the number of formulas in this chain that begin with the negation sign is even, and negative if this number is odd. A sentence S is said to imply a sentence T whenever T holds in every model for which S holds. The following theorem is established: Let S and T be sentences such that S implies T; then there exists a sentence M such that S implies M and M implies T, and that a relation symbol has positive [negative] occurrences in M only if it has positive [negative] occurrences in both S and T.—This interpolation theorem is an important extension of Craig's Lemma, which results if the distinction between positive and negative occurrences is suppressed.

E. W. Beth (Amsterdam)

5556:

Blanché, Robert. Sur la structuration du tableau des connectifs interpropositionnels binaires. J. Symb. Logic **22** (1957), 17–18.

The 'square of quaternality' [see W. H. Gottschalk, same J. **18** (1953), 193–196; MR **15**, 494] is here extended to a diagram, with the traditional symmetries, whose vertices represent the ten strictly binary connectives of the propositional calculus. This diagram consists of two hexagons, with each of a pair of opposite edges of the one identified with a corresponding edge of the other. The author has discussed this matter further in Theoria, Lund **19** (1953), 89–130. *R. C. Lyndon* (Ann Arbor, Mich.)

5557:

Hailperin, Theodore. A theory of restricted quantification. II. J. Symb. Logic **22** (1957), 113–129.

[For part I, see same J. **22** (1957), 19–35; MR **19**, 626.] The use of free variables with restricted ranges is interpreted in a theory $\mathcal{QF}\nu$ and natural deduction methods are introduced. Further, definite descriptions and the Hilbert ϵ-symbol are discussed. The author argues that his formalism $\mathcal{QF}\nu$ gives a simpler treatment of such rules as existential instantiation. *Dana Scott* (Chicago, Ill.)

5558:

Słupecki, Jerzy. On some partial systems of the propositional calculus. Studia Logica **8** (1958), 177–187. (Polish. Russian and English summaries)

This is a generalization of the reviewer's paper [Studia Logica **3** (1955), 208–226; MR **17**, 226]. Let S_c be the system of the classical implicative propositional calculus (the implication sign is the only connective) and let S_n $(n=2, 3, \cdots)$ be the subset of the set of all formulas α in the language of S_c satisfying the condition (n): the number of occurrences of any propositional variable in α is divisible by n. The author deals with the systems $S_n{}^*$ $(n=2, 3, \cdots)$ which are the intersection of S_c and S_n. Each system $S_n{}^*$ may also be characterized by means of a finite and normal matrix $\mathfrak{M}_n{}^*$ being the product of the two-valued matrix for the classical implication and of the matrix $\mathfrak{M}_n$ adequate for formulas satisfying the condition (n). It is proved that the systems S_n and $S_n{}^*$ are axiomatizable and that the set of axioms for $S_n{}^*$ is also a set of axioms for the classical propositional calculus with quantifiers. The negation is then defined as follows: $Np \overset{\mathrm{df}}{=} Cp \prod_p p$. Some results concerning the degree of completeness of S_n and $S_n{}^*$ are also presented. *H. Rasiowa* (New Orleans, La.)

5559:

Rose, Alan. Nouvelle méthode pour déterminer les formules qui correspondent à des éléments universels de décision. C. R. Acad. Sci. Paris **249** (1959), 870–872.

The paper presents, for the calculation of universal decision elements, a mechanical method simpler than that of J. M. Pugmire and A. Rose [Z. Math. Logik Grundlagen Math. **4** (1958), 1–9; MR **20** #4481].

R. M. Baer (Berkeley, Calif.)

5560:

Zlot, William Leonard. Some comments on the role of the axiom of choice in the development of abstract set theory. Math. Mag. **32** (1958/59), 115–122.

A short expository article on the role of the axiom of choice in set theory. *O. Frink* (University Park, Pa.)

5561:

Mendelson, Elliott. The Axiom of Fundierung and the axiom of choice. Arch. Math. Logik Grundlagenforsch. **4** (1958), 65–70.

In his paper J. Symb. Logic **21** (1956), 350–366 [MR **18**, 864], the author showed that if Gödel's axioms A, B, C are consistent, then one cannot prove from them the following weak axiom of choice H_1: Every denumerable set of disjoint unordered pairs has a choice set. The present paper extends these results to show that if A, B, C are consistent, then neither H_1 nor axiom D is provable from A, B, C and the Fundierungsaxiom (in the form stating that there are no infinite descendent ε-sequences). The proof is similar to that of the previous paper, where the role of the "Urelemente" of Fraenkel and Mostowski [cf. A. Fraenkel, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl. **1922**, 253–257; A. Mostowski, Fund. Math. **31** (1939), 201–252] was taken by disjoint infinite descending ε-sequences. Here in their place we have a sequence of disjoint sets $q_1, q_2, q_3, \cdots$ such that, for any integer n, q_n is a denumerable transitive set (i.e., if $u \in q_n$, then $u \subseteq q_n$) in which ε is a total, irreflexive, dense order without first or last element.

L. N. Gál (New Haven, Conn.)

5562:

Wegel, Herbert. Axiomatische Mengenlehre ohne Elemente von Mengen. Math. Ann. **131** (1956), 435–462.

A first order system of 13 axioms (some of them rather complicated) of a set theory is given concerning the following four primitive notions: (1) the unary predicate of being a set; (2) the unary predicate of being a function (of exactly four set-variables, with one set-value); (3) the binary inclusion (i.e., semi-ordering) predicate (between sets); (4) the six-placed predicate for five sets, say a, b, c, d, e and for one function, say φ, satisfied if and only if $\varphi(a, b, c, d) = e$. The membership-predicate is not definable (as asserted without proof), but partly replaced by the inclusion together with the (obviously defined) notion of a "singleton". Otherwise, the author's system seems to be intentionally near to that of v. Neumann [cf. Math. Z. **27** (1928), 669–752] and includes some ideas of Bernays [cf. J. Symb. Logic. **2** (1937), 65–77] and of Fraenkel [cf. Math. Z. **22** (1925), 250–273].

The reviewer finds it difficult to detect and to judge the author's intentions and results exactly, because of a great number of nonintuitive and complicated notions introduced by the author, some of them (as in the crucial def. 7) being metamathematical but not separated from the mathematical ones. The author himself stresses the "extraordinary weakness" (ausserordentliche Schwäche) of his system as an advantage (with respect to the consistency problem). He asserts that his axioms I–XIII are theorems of the system of Fraenkel and he formulates and infers some basic set-theoretical theorems (e.g., that of Bernstein and that on the power of the potency-set). On the other hand, he states (without proof; see the bottom of p. 461) that the notion of a set-sum is not definable in his system.

L. Rieger (Prague)

5563:

Shoenfield, J. R. On the independence of the axiom of constructibility. Amer. J. Math. **81** (1959), 537–540.

Let GCH' be the so-called weakened generalized continuum hypothesis (there is an ordinal α_0 such that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every $\alpha \geqq \alpha_0$) and let ACon′ be the so-called weakened Gödel's axiom of constructibility [stating that every set of integers is constructible; for these notions see K. Gödel, *The consistency of the continuum hypothesis*, Princeton Univ. Press, 1940; MR **2**, 66]. The author proves the following Theorem 1: If the axiom of constructibility is not provable from axioms A–D, then it is not provable from axioms A–D, the axiom of choice and GCH′.—Theorem 2: If ACon′ is not provable from axioms A–C, then it is not provable from axioms A–D, the axiom of choice, and the generalized continuum hypothesis (and hence the axiom of constructibility is not provable from these axioms).

The method is based on a modification of Gödel's construction of the model Δ by means of a ninth basic operation $\mathfrak{F}_9(X, Y) = a \cap X$, where a is a constant nonconstructible set consisting of constructible elements. [For this method cf. Hajnal, Z. Math. Logik Grundlagen Math. **2** (1956), 131–136; MR **19**, 1031]. *L. Rieger* (Prague)

5564:

Rabin, Michael O. Arithmetical extensions with prescribed cardinality. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 439–446.

A system $\mathfrak{R} = \langle A, R_0, \cdots, R_\alpha, \cdots \rangle_{\alpha<\rho}$, where all R_α are relations on the set A or all R_α are functions of zero or more variables on A, is called complete if all relations or all functions on A appear among the R_α. If all the R_α are functions, the system is called an algebra. The cardinality of $\mathfrak{R}$ is the cardinality of A. There is a corresponding first-order theory with predicate letters or function letters for each R_α and in which the valid formulas are the formulas true in R. Answering a question of Tarski and Vaught [Compositio Math. **13** (1957), 81–102; MR **20** #1627] the author proves: (1) If $m^{\aleph_0} = m$ then every complete algebra (and hence every relational system) of cardinality m has a proper arithmetically equivalent extension of the same cardinality; (2) If the cardinality m of a complete system is less than the first weakly inaccessible number and if $m^{\aleph_0} > m$, then it has no proper arithmetically equivalent extension of the same cardinality. (For $m > \aleph_0$, the generalized continuum hypothesis is used to prove (2).) In showing (1), the author derives a theorem which gives a useful generalization of the method used by Skolem to construct non-standard models for number theory.

E. Mendelson (New York, N.Y.)

5565:

Moh, Shaw-kwei. The modal systems and implication systems. Acta Math. Sinica **9** (1959), 121–142. (Chinese. English summary)

Add the necessity operator $\Box$ to the two-valued system and define Fpq as $\Box Cpq$, Gpq as $KFpqFqp$. Construe p, q, etc., as schemata and dispense with the rule of substitution. The basic modal system A which is contained in most modal systems is axiomatized by: (1) $\Box p$ is a theorem of A, if p is a theorem of the two-valued logic; (2) Fpq, $p \to q$; (3) $F\Box pp$; (4) Fpq, $\Box p \to \Box q$; (5) p, $q \to Kpq$; (6) $Gpq \to G\Box p\Box q$. The following additional axioms and rules are considered: (7) $Fpq \to G\Box p\Box q$; (8) $CFpqC\Box p\Box q$; (9) $FFpqC\Box p\Box q$; (10) $Fpq \to F\Box p\Box q$; (11) $CFpqF\Box p\Box q$; (12) $FFpq\Box p\Box q$; (13) $CGpqG\Box p\Box q$; (14) $FGpqG\Box p\Box q$;

(15) $p \rightarrow FqKpq$; (16) $p \rightarrow FFpqq$; (17) $CpFFpqq$; (18) $FpFFpqq$. Nine systems are obtained by adding to A: (7), (8), (9), (10), (8) and (10), (9) and (10), (11), (9) and (11), (12), respectively. Each of the ten systems can again be extended by adding any one of the statements (13) to (18). In this way, hierarchies of systems are obtained which serve as a framework for classifying usual modal systems. Leaving out two-valued systems, there are at most 65 equivalent classes. In particular, $S2$, $S3$, $S4$ of Lewis all occur among these systems. Improvements of the basic calculi of Shen [same Acta **7** (1957), 132–142; MR **20** #3776] are suggested. All these systems are subject to the paradoxes of implication such as $\Box p \rightarrow Fqp$. To avoid these paradoxes, F is taken as an additional primitive. The author gives two such systems closely related to system A. *Hao Wang* (Murray Hill, N.J.)

5566:

Šanin, N. A. Über einen Algorithmus zur konstruktiven Dechiffrierung mathematischer Urteile. Z. Math. Logik Grundlagen Math. **4** (1958), 293–303. (Russian. German summary)

The constructive interpretation of mathematical propositions, given by the author [Trudy Mat. Inst. Steklov. **52** (1958), 226–311; MR **21** #2] is illustrated here by some examples. *A. Heyting* (Amsterdam)

5567:

★**Kalmár, László. An argument against the plausibility of Church's thesis.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 72–80. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii+297 pp. $8.00.

The author asserts (1) that the notion of an effective rule must remain "pre-mathematical" and does not permit the restrictions imposed by an exact mathematical definition and (2) that Church's thesis (C: to every effective rule for computing a sequence of natural numbers there exists a set of recursive equations with number e such that the function defined by the rule has the same values as $U[\tilde{\mu}_y T(e, x, y)]$ in Kleene's notation [*Introduction to metamathematics*, Van Nostrand, New York, 1952; MR **14**, 525]) is too restrictive. He argues (2′) that C implies the existence of an absolutely undecidable problem of the form $(x)A(x)$, specifically $(x)\neg T(e, e, x)$ for some e. No reasons are given for (1), which is explicitly denied by Heyting [Dialectica **12** (1958), 332–344; MR **20** #6360; p. 342] and implicitly by Gödel [ibid. **12** (1958), 280–287; MR **21** #1275; footnote 2 on p. 283]. In fact, (1) contradicts the very basis of intuitionist mathematics where the notion of construction is taken as a primitive (mathematical) notion, C taking the form of a mathematical statement

$$(\alpha)(Ee)(E\beta)(x)\{\alpha(x) = U[\beta(x)] \ \&\ T[e, x, \beta(x)]\}$$

with α ranging over constructive number theoretic functions (not, of course, free choice sequences). While it is doubtful whether C can be decided on the properties of constructive functions and constructive proofs so far made explicit in Heyting's systems or even in Brouwer's argument for the fan theorem, there is no doubt that the author's step from (2) to (2′) is defective on the intuitionistic interpretation of the logical terms involved; and since he is concerned with decidability and not merely truth, this is the appropriate interpretation. He considers the species $R(x, y)$, short for

$$\begin{aligned}\{(Ez)T(x, x, z) \rightarrow (Ev)[T(x, x, v) \\ \&\ (w)_{<v} \neg T(x, x, w) \ \&\ U(v) = y]\} \\ \&\ \{\neg (Ez)T(x, x, z) \rightarrow y = 0\}\end{aligned}$$

(i.e., $y = U[\mu_y T(x, x, y)]$ with Hilbert's μ-symbol). On the assumption that $R(x, y)$ does not have functional character, i.e., $\neg(x)(Ey)R(x, y)$, he concludes (i) $(Ex)\neg(Ey)R(x, y)$ and (ii) if $\neg(Ey)R(x_0, y)$ then the problem $(z)\neg T(x_0, x_0, z)$ is absolutely undecidable, i.e., $\neg[(z)\neg T(x_0, x_0, z) \vee \neg(z)\neg T(x_0, x_0, z)]$. Now (ii) is correct and, in fact $\neg\neg(Ey)R(x, y)$ is easily proved in intuitionistic first order arithmetic both on the assumption of $(Ez)T(x, x, z)$ and of $\neg(Ez)T(x, x, z)$, hence from $\neg\neg[(Ez)T(x, x, z) \vee \neg(Ez)T(x, x, z)]$ and so outright. But (i) is certainly not derivable in the latter, as seen by Kleene's recursive realizability, and is in fact quite implausible. The author emphasizes at length (pp. 77–79) the fact that if, for a primitive recursive A, $(x)A(x)$ is absolutely undecidable then so is a statement of the form: the formula '$\mathfrak{A}$' is consistent over the predicate calculus. This is a familiar consequence of Gödel's reduction of the decision problem for the class of such $(x)A(x)$ to the decision problem for the predicate calculus [Monatsh. Math. Phys., **38** (1931), 173–198; p. 194] (which, together with C, leads to the recursive undecidability of the predicate calculus). In the reviewer's opinion, it should be possible to discuss C fruitfully within a suitable mathematical framework. In particular, it would seem naive to be satisfied with the so-called 'heuristic' evidence; for, before Gödel's completeness and incompleteness theorems one had comparable evidence for (i) the (true) assertion that all logically valid statements are derivable in the usual predicate calculus, and (ii) the (false) assertion that if a number theoretic theorem can be proved at all it can be proved formally in Principia Mathematica. It is to be noted that Gödel decided (i) and (ii) without an 'exact' definition of the notions of validity and number theoretic truth, but on the basis of certain elementary properties of these notions. An exact definition of these notions was given only later by Tarski (granted that the set theoretical definition is regarded as 'exact'). This shows that the alleged impossibility of establishing results for notions at a 'pre-mathematical' stage is not well-founded (nor is the sense of 'pre-mathematical'). It is of course not surprising that C cannot be formulated in that part of mathematics which confines itself to purely model theoretic (classical) concepts since they are concerned with truth and not decidability. But also C is quite unimportant here: 'rejecting' C merely means replacing 'effective' by 'recursive' in all statements, the theory of recursive functions having a sufficiently interesting mathematical structure in its own right; and 'accepting' C does not curtail the subject since it studies non-recursive concepts anyway (e.g. classifying them in hierarchies). *G. Kreisel* (Reading)

5568:

★**Kreisel, Georg. Interpretation of analysis by means of constructive functionals of finite types.** Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 101–128. Studies in Logic and the Foundations of Mathematics.

North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. \$8.00.

The author describes an interpretation, due to Gödel, but not published before, of intuitionistic arithmetic (A) to a quantifier-free system (P) in which primitive recursive functionals of arbitrary finite type can be defined. By an inductive process there is associated with every formula $\mathscr{A}$ of (A) a formula

$$(E\varphi_1)\cdots(E\varphi_m)(\psi_1)\cdots(\psi_n)A(\varphi_1, \cdots, \varphi_m, \psi_1, \cdots, \psi_n),$$

abbreviated $(E\varphi)(\psi)A(\varphi, \psi)$. For instance, if $(E\varphi)(\psi)A(\varphi, \psi)$ is associated with $\mathscr{A}$ and $(E\varphi_1)(\psi_1)B(\varphi_1, \psi_1)$ with $\mathscr{B}$, then

$$(E\Phi_1)(E\Psi)(\varphi)(\psi_1)\{A(\varphi, \Psi(\varphi, \psi_1)) \rightarrow B(\Phi_1(\varphi), \psi_1)\}$$

is associated with $\mathscr{A}\rightarrow\mathscr{B}$. Gödel has shown that if $\mathscr{A}$ is provable in (A), then there is a primitive recursive φ_0 such that $A(\varphi_0, \psi)$ is provable in (P).

In analysis φ and ψ must range over functionals of higher type. In order to define continuous functionals of arbitrary finite type, the author introduces by an inductive definition neighbourhoods $U^{(i)}$ of type i; if $\mathscr{F}_{i+1}$ $(i=1, \cdots, n)$ is a class of functions, defined on $U^{(i)}$, then the notions "The function $\tau^{(n)}$ defined on $U^{(n-1)}$ is in $C(\mathscr{F}_1, \cdots, \mathscr{F}_n)$" and "$\tau^{(n)} \in U^{(n)}$" are defined by induction. Intuitively, $U^{(n)}$ is the set of functions which take given values on a finite set of neighbourhoods $U^{(n-1)}$; $C(\mathscr{F}_1, \cdots, \mathscr{F}_{n+1})$ is the set of functions in $\mathscr{F}_{n+1}$, defined on neighbourhoods of functions in $C(\mathscr{F}_1, \cdots, \mathscr{F}_n)$. A functional is an equivalence class of functions; two functions are equivalent if they are equal when both are defined.

$C(\mathscr{F}_1, \cdots, \mathscr{F}_{n+1})$ is called the class of (arbitrary) continuous functionals if each $\mathscr{F}_i$ consists of all functions on $U^{(i-1)}$; it is called the class of recursive continuous functionals if $\mathscr{F}_i$ $(i=1, \cdots, n)$ consists of all functions on $U^{(i-1)}$, but $\mathscr{F}_{n+1}$ of the recursive functions on $U^{(n)}$. The formula $(E\varphi)(\psi)A(\varphi, \psi)$, associated to a formula $\mathscr{A}$ of analysis, is denoted by $\mathscr{A}_3'$ if ψ ranges over continuous and φ over recursively continuous functionals. If $\mathscr{A}$ contains neither existence nor disjunction symbols, then $\mathscr{A}\leftrightarrow\mathscr{A}_3'$. If $\mathscr{A}$ does not contain universal function quantifiers, then $\neg\mathscr{A}\leftrightarrow(\neg\mathscr{A})_3'$. The proofs are classical; they rest upon the following theorem. If, for continuous $\tau^{(n)}$, $\sigma^{(n)}$, $(\tau^{(n)})(E\sigma^{(n)})A(\tau^{(n)}, \sigma^{(n)})$, then there exists a recursively continuous Ψ such that $(\tau^{(n)})A(\tau^{(n)}, \Psi(\tau^{(n)}))$. This in its turn is based on the "density theorem": If $\mathscr{F}_i$ includes all primitive recursive functions of type i, then there is a recursive, recursively dense subset of the functions $\tau^{(n)}$ in $C(\mathscr{F}_1, \cdots, \mathscr{F}_n)$. The last two theorems are proved intuitionistically. The theory sketched above is compared with two other theories. $\mathscr{A}_1'$ denotes the associated formula $(E\varphi)(\psi)A(\varphi, \psi)$, where φ and ψ range over effective operations, as defined for type 2 by Myhill and Shepherdson [Z. Math. Logik Grundlagen Math. **1** (1955), 310–317; MR **17**, 1039], here generalized to higher types. $\mathscr{A}_2'$ denotes the associated formula, where ψ ranges over arbitrary functionals and φ over Kleene's general recursive functionals [Trans. Amer. Math. Soc. **91** (1959), 1–52; MR **21** #1273]. The above results do not hold in these interpretations. The author argues that they are intuitively unsatisfying. The first is restricted to recursive arguments, while recursively continuous functions of arbitrary arguments are constructively meaningful; in the second calculations of infinite length are admitted. However, $\mathscr{A}_2'$ is of interest for independence proofs.

A. Heyting (Amsterdam)

5569:

★Rasiowa, H. Algebraische Charakterisierung der intuitionistischen Logik mit starker Negation. Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting), pp. 234–240. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. viii + 297 pp. \$8.00.

A. A. Markov [Uspehi Mat. Nauk **5** (1950), no. 3 (37), 187–188] has considered a system of intuitionistic number theory, in which, besides the intuitionistic negation $\neg$, there occurs a strong negation $\sim$. Vorob'ev [Dokl. Akad. Nauk SSSR **85** (1952), 465–468, 689–692; MR **14**, 233] gave a system of axioms for the underlying propositional calculus S. In this paper the notion of an N-lattice $\mathscr{M} = \langle A, e, +, \cdot, \sim, \rightarrow, \neg \rangle$ is introduced by a system of axioms. $\mathscr{M}$ is called regular, if (i) and (ii) are valid: (i) whenever $e\rightarrow x=e$, then $x=e$; (ii) whenever $a\rightarrow b=e$, $b\rightarrow a=e$, $\sim a\rightarrow\sim b=e$, $\sim b\rightarrow\sim a=e$, then $a=b$. A regular matrix satisfies S if and only if it is an N-lattice. In order to give a topological interpretation of S, the notion of a quasifield of sets is introduced. This is a set $B(\mathscr{X})$ of subsets of a set $\mathscr{X}$, which contains $\mathscr{X}$ and which is closed with respect to union, intersection and the operation $\sim X = \mathscr{X} - g(X)$; here $g(x)$ is a one-to-one mapping of $\mathscr{X}$ onto $\mathscr{X}$ such that $g(g(x))=x$. For the case that $B(\mathscr{X})$ is a set of open subsets of a T_0-space $\mathscr{X}$, sufficient conditions are given for $B(\mathscr{X})$ being an N-lattice. Every N-lattice is isomorphic to such a $B(\mathscr{X})$. Proofs are briefly sketched.

A. Heyting (Amsterdam)

5570:

Goodstein, R. L. Models of propositional calculi in recursive arithmetic. Math. Scand. **6** (1958), 293–296.

This paper is concerned with models of Post's $(N+1)$-valued logics within recursive arithmetic. Let the truth-values be $0, \cdots, N$ with 0 as designated value. The truth-functions $p \vee q$, $p \,\&\, q$, $\sim p$ and $p\rightarrow q$ can be represented by the recursive functions $p \dot{-} (p \dot{-} q)$, $p + (q \dot{-} p)$, $\{1 \dot{-} (1 \dot{-} (N \dot{-} p))\}(p+1)$ and $(N \dot{-} p) \dot{-} (N \dot{-} q)$ respectively. It is proved that the $(N+1)$-valued analogue of the tertium-non-datur has a representing function which is identically zero. Moreover, the schemata of modus ponens and mathematical induction hold in this model. This, the author states, "shows that, by means of the model, many-valued logics have the same power to reveal mathematical connections as classical logic."

The substitution theorem $x=y\rightarrow\{f(x)=f(y)\}$, where $\rightarrow$ denotes the $(N+1)$-valued implication connective, fails, however, for $N>1$ and so does the deduction theorem. The author concludes his paper by supplying a model for a certain propositional calculus with denumerably many truth-values.

J. C. E. Dekker (New Brunswick, N.J.)

5571:

Klaua, Dieter. Die Präzisierung des Berechenbarkeitsbegriffes in der Analysis mit Hilfe rationaler Funktionale. Z. Math. Logik Grundlagen Math. **5** (1959), 33–96.

The author develops here a computable analysis, defining a number of important concepts with the aid of the notion of "rational functional" which has its origins in Grzegorczyk, Fund. Math. **42** (1955), 168–202 [MR **19**, 238], and which allows the definition of computable real

functions without reference to number-theoretic recursiveness. A number of alternate definitions are provided for several notions, and in certain cases their equivalence is demonstrated. It is shown that the definitions of computable real functions are equivalent when the functions are computably continuous over computable intervals. However, under Klaua's definition of computable real function using rational functionals, there exist such functions which are discontinuous, a contrast to a property of such functions under the definition in the paper cited above. Differentiation and integration are defined and it is shown that every monotone and computable uniformly continuous function is integrable, and that the fundamental theorem holds. Section titles are as follows: 1. Computable rational sequences; 2. Computable rational functions and relations; 3. Computable rational functionals and relationals; 4. Computable real numbers and sequences; 5. Computable real functions and relations; 6. Continuous computable real functions; 7. Computable continuity; 8. Computable differentiation; 9. Computable integration. *E. J. Cogan* (Bronxville, N.Y.)

5572:

Lacombe, Daniel. Les ensembles récursivement ouverts ou fermés, et leurs applications à l'analyse récursive. C. R. Acad. Sci. Paris **245** (1957), 1040–1043; **246** (1958), 28–31.

To avoid misunderstandings (since the usage in the literature is anything but uniform) we recall from same C. R. **240** (1955), 2478–2480; **241** (1955), 13–14, 151–153 [MR **17**, 225] the author's definitions of recursive real number and recursive real function. Let R be the real numbers, including $\pm\infty$. Let $\mathfrak{B}$ be the family of all number-theoretic partial functions. Let $\{\rho_i\}$ be any of the usual effective enumerations of the rationals. $\alpha\in R$ is called recursive if $\{i\mid\rho_i<\alpha\}$ is recursive. $f\in\mathfrak{B}$ is called an approximation if $f(i)\leqq 1$ and there exist $\alpha,\beta\in R$ for which $\alpha\leqq\beta$ and $f(i)$ is 0, undefined or 1 according as $\rho_i<\alpha$, $\alpha<\rho_i<\beta$ or $\beta<\rho_i$. If $\alpha=\beta$, f is called a perfect approximation to α (so that there are either three perfect approximations to α or one, according as α is rational or irrational). Let $E\subset R$. Then $\phi: E\rightarrow R$ is called recursive if some (consistent) partial recursive operator $\Phi:\mathfrak{B}\rightarrow\mathfrak{B}$ maps approximations into approximations and perfect approximations to $\alpha\in E$ into perfect approximations to $\phi(\alpha)$.

In the papers under review the author calls $E\subset[0,1]$ recursively open if it is the union of a r.e. (i.e., recursively enumerable; equivalently, of a recursive) sequence of rational open intervals (i.e., open in [0, 1]). The following theorems are proved in outline. The collection of all recursively open sets is closed under finite intersection and recursive infinite union. $E\subset[0,1]$ is recursively open if and only if it is open (in [0, 1]) and the collection of all rational closed intervals contained in E is r.e. $E\subset[0,1]$ is recursively closed (i.e., its complement in [0, 1] is recursively open) if and only if it is the set of all points at which some recursive function ϕ: $[0,1]\rightarrow R$ assumes its maximum. An isolated point of a recursively closed set is recursive. [This is not true in the space of all (full) number-theoretic functions with the product topology; cf. Kuznecov and Trahtenbrot, Dokl. Akad. Nauk SSSR **105** (1955), 897–900; MR **17**, 1039; remark after theorem 6′.] There is a recursive real function ϕ: $[0,1]\rightarrow R$ whose maximum is not attained at any recursive point. [This theorem was announced without proof by the author at the end of C. R. Acad. Sci. Paris **241** (1955), 1252–1254; MR **17**, 701; it is easy to see that the maximum itself must be recursive.] *J. Myhill* (Berkeley, Calif.)

5573:

Šanin, N. A. Über konstruktive lineare Funktionale in einem konstruktiven Hilbertschen Raum. Z. Math. Logik Grundlagen Math. **5** (1959), 1–8. (Russian. German summary)

A point of constructive Hilbert space l_2 is a recursive, recursively convergent sequence, the elements of which are finite sequences of rational numbers. The square of the norm $N\gamma$ of a point γ is defined by a recursive sequence of rational numbers, the Gödel number of which is a partial recursive function of the Gödel number of the defining sequence of γ. The same is valid for the scalar product $C(\gamma_1,\gamma_2)$. A linear functional ϕ in l_2 is called locally bounded if $|\phi(\gamma)|\leqq\lambda N\gamma$ for every γ in l_2. It is called normed if its norm $\sup_{N\gamma\leqq 1}|\phi(\gamma)|$ is a recursive real number. In the classical theory, by Riesz' theorem, every locally bounded linear functional is of the form $C(\gamma_0,\gamma)$, and is therefore normed with norm $N\gamma_0$. By means of Specker's example of a monotonous recursive sequence of rational numbers without a recursive limit [J. Symb. Logic **14** (1949), 145–158; MR **11**, 151], a locally bounded linear functional is constructed which is not normed. It follows that Riesz' theorem is not valid in the constructive sense. It is further shown that the following three properties of a locally bounded linear functional are equivalent: I. $\phi(\gamma)$ is of the form $C(\gamma_0,\gamma)$, $\gamma_0\neq 0$; II. there exists a point γ_1, $\gamma_1\neq 0$, which is orthogonal to every point of the subspace $\phi(\gamma)=0$; III. ϕ is normed and there exists a point γ_2 so that $\phi(\gamma_2)\neq 0$. Riesz' theorem is valid in the form: If the norm of ϕ is either $\neq 0$ or $=0$, then there exists a γ_0 so that $\phi(\gamma)=C(\gamma_0,\gamma)$. The results are valid as well for an axiomatically defined constructive Hilbert space.

A. Heyting (Amsterdam)

5574:

Skolem, Th. Remarks on the connection between intuitionistic logic and a certain class of lattices. Math. Scand. **6** (1958), 231–236.

In a remarkable paper published in 1919 under the title "Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen" [Skr. Utgitt Vidensk. Kristiania. I. Mat.-Nat. Kl. **1919**, no. 3] the author studied several questions about distributive lattices in a completely abstract and general form. At that time, however, he did not use the terminology we know today, and he drew his lattice diagrams sideways (with the smaller elements to the left, a natural generalization of diagrams for linear orderings) instead of employing the current and correct top-bottom format. Hence, his work escaped the notice that it actually deserves. In the present article the author applies what he has known for such a long time in giving a lattice-theoretic interpretation of the Intuitionistic Propositional Calculus of Heyting and in establishing the known result that if a disjunction is provable, then at least one of the terms is provable. Nearly the same proof and exactly the same kind of interpretation is contained in, say, the two papers of J. C. C. McKinsey and A. Tarski [Ann. of Math. (2) **47** (1946),

122–161; J. Symb. Logic **13** (1948), 1–15; MR **7**, 359; **9**, 486].

These authors were quite aware of Skolem's early work; they refer to it in the first of the above-mentioned papers and make use of it. {The decision method ascribed to the reviewer is actually due to S. Jaśkowski. A reference and a proof may be found in a paper by G. F. Rose [Trans. Amer. Math. Soc. **75** (1953), 1–19; MR **13**, 1].}

Dana Scott (Chicago, Ill.)

5575:

★Bindel, Ernst. Die geistigen Grundlagen der Zahlen. Verlag Freies Geistesleben, Stuttgart, 1958. 258 pp. DM 19.80.

5576:

★Dedekind, Richard. Was sind und was sollen die Zahlen? 8te unveränderte Aufl. Friedr. Vieweg & Sohn, Braunschweig, 1960. xii+47 pp. Brosch: DM 3.80.

5577:

Felscher, Walter; und Schmidt, Jürgen. Natürliche Zahlen, Ordnung, Nachfolge. Arch. Math. Logik Grundlagenforsch. **4** (1958), 81–94.

"Beim Aufbau der Arithmetik vom Peanoschen Axiomensystem her entsteht jedesmal das Problem, die Größenanordnung der natürlichen Zahlen zu definieren und als eine Wohlordnung nachzuweisen. Im folgenden geben wir einen streng relationentheoretischen Weg an, der uns neben dem Vorteil der Natürlichkeit auch den nicht mehr zu übertreffender Kürze zu bieten scheint. Zum Vergleich reproduzieren wir den klassischen Vorgang von Dedekind und analysieren seinen Zusammenhang mit unserem direkteren Verfahren. Schließlich stellen wir, auf dem von uns eingeschlagenen Wege, zwischen dem auf der Nachfolge beruhenden Peanoschen und dem ordnungstheoretischen Axiomensystem für die natürlichen Zahlen volle Äquivalenz her und betten diese in einen größeren ordnungstheoretischen Zusammenhang ein." (Authors' summary.)

Dana Scott (Chicago, Ill.)

SET THEORY

See also 5560, 5564, 5577, 5586, 5587, 5649.

5578:

Riss, J. Les théorèmes de Zorn et de Zermelo. Publ. Sci. Univ. Alger. Sér. A **3** (1956), 121–124.

The author gives fairly simple proofs of Zorn's lemma and of the well-ordering theorem, assuming the axiom of choice. The proofs use the notion of a strictly increasing correspondence between well-ordered sets. They are somewhat similar to a proof given by Bourbaki. There are no references to other proofs.

O. Frink (University Park, Pa.)

5579:

Eyraud, Henri. Sur les suites de fonctions premières. Ann. Univ. Lyon. Sect. A (3) **21** (1958), 5–17.

5580:

Banaschewski, B. On transfinite iteration. Fund. Math. **46** (1959), 225–229.

Let $\mathfrak{D}$ be a class of sets closed under the union of chains, and let $f:\mathfrak{D}\to\mathfrak{D}$ be an operator such that $X\subseteq f(X)\subseteq f(Y)$ whenever, $X, Y\in\mathfrak{D}$ and $X\subseteq Y$. Define the transfinite powers of $f^\alpha:\mathfrak{D}\to\mathfrak{D}$ for each ordinal α by the recursion equation $f^\alpha(X)=X\cup\bigcup\{f(f^\beta(X)):\beta<\alpha\}$. {This definition differs slightly from the author's at limit numbers. The author's definition also was wrong (in a trivial way) for the case $\alpha=0$.} Question: for which ordinals α will $f^\alpha=f^{\alpha+1}$ (and $f^\alpha=f^\gamma$ for all $\gamma\geq\alpha$)? An obvious answer would take α to be the first ordinal whose cardinal is greater than the cardinality of $\mathfrak{D}$. Some less obvious criteria are given in this note. Sample: Let α be the first ordinal whose cardinal $\mathfrak{k}$ is regular and has the property that every chain $\mathfrak{C}\subseteq\mathfrak{D}$ of power $\mathfrak{k}$ which is closed under unions of subchains of power strictly less than $\mathfrak{k}$ satisfies the equation $f(\bigcup\mathfrak{C})=\bigcup\{f(X):X\in\mathfrak{C}\}$. This condition and some others are compared with a criterion given by G. Schwartz [J. Symb. Logic **21** (1956), 265–266; MR **18**, 456] and the results are extended in an interesting way to transfinite compositions of several such operators.

Dana Scott (Chicago, Ill.)

5581:

Kurepa, G. General continuum hypothesis and ramifications. Fund. Math. **47** (1959), 29–33.

The main theorem states that for any ordinal α, the following are equivalent. (1) $2^{\aleph_\alpha}=\aleph_{\alpha+1}$. (2) Let the set of all sequences of type $\leq\omega_{\alpha+2}$ whose terms are ordinals $<\omega_\alpha$ be partially ordered by 'is an initial segment of'; then in this set, any initial portion of length $\omega_{\alpha+2}$ in which every chain has at most $\aleph_\alpha$ nonzero terms contains a sequence of type $\omega_{\alpha+2}$. The theorem is related to the author's earlier work about Souslin's problem.

L. Gillman (Princeton, N.J.)

COMBINATORIAL ANALYSIS

See also 5034.

5582:

Skolem, Th. Some remarks on the triple systems of Steiner. Math. Scand. **6** (1958), 273–280.

The author gives various methods for constructing triples of elements from a given set in such a way that each pair of distinct elements of the set is contained in just one triple. One of the methods is related to his result [Math. Scand. **5** (1957), 57–68; MR **19**, 1159] that if $n\equiv 0$ or 1 (mod 4) the numbers $1, 2, \cdots, 2n$ can be placed in n disjoint pairs $(a_1, b_1), \cdots, (a_n, b_n)$ such that $b_r=a_r+r$ for $r=1, \cdots, n$. He conjectures that if $n\equiv 2$ or 3 (mod 4) the numbers $1, 2, \cdots, 2n-1, 2n+1$ can be placed in n disjoint pairs with the same property.

H. Davenport (Cambridge, England)

5583:

Norman, Robert Z.; and Rabin, Michael O. An algorithm for a minimum cover of a graph. Proc. Amer. Math. Soc. **10** (1959), 315–319.

A set C of edges of a graph G is a "cover" [J. P. Roth, Trans. Amer. Math. Soc. **88** (1958), 301–326; MR **20** #3755]

if every vertex of G lies on an edge of C. Problem: Devise an algorithm which computes for any graph a cover with minimum cardinality. Algorithms are given which compute such a minimum and compute the set of all such minima. A "matching" [C. Berge, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 842–844; MR **20** #1323] is a set of pairwise disjoint edges. Problem: Find a matching of maximum cardinality. Subalgorithms are added to the original, converting a minimum cover into a maximal matching, and conversely. *J. P. Roth* (Yorktown Heights, N.Y.)

ORDER, LATTICES

See also 5574, 5577, 5685, 5878.

5584:

Čulík, Karel. Über die lexikographische Summe der teilweise geordneten Mengen. Časopis Pěst. Mat. **84** (1959), 16–30. (Czech. Russian and German summaries)

Es sei M eine teilweise geordnete (t.g.) Menge. Eine t.g. Teilmenge $(\emptyset\neq)\ P\subset M$ heisst eingelegt in M, wenn aus $x, y\in P$, $z\in M-P$ die Beziehungen $x<z\Leftrightarrow y<z$ und $z<x\Leftrightarrow z<y$ folgen. Sind die Elemente einer Zerlegung $\bar{M}$ der Menge M t.g. und eingelegt in M, so wird die im Sinne der Formeln: $\bar{a}, \bar{b}\in\bar{M}$, $\bar{a}\neq\bar{b}$, $x\in\bar{a}$, $y\in\bar{b}\Rightarrow\{\bar{a}<\bar{b}\Leftrightarrow x<y\}$ t.g. Menge $\bar{M}$ als eine Faktormenge von M bezeichnet.— In der vorgelegten Arbeit wird die Möglichkeit einer Darstellung von M als die lexikographische Summe eines Systems von t.g. Mengen $M_u\neq\emptyset$ über einer t.g. Menge $N\neq\emptyset$, sowie andere ähnliche Fragestellungen, behandelt. Insbesondere wird gezeigt, dass die erwähnte Darstellung $M=\sum_N M_u$ genau dann möglich ist, wenn M_u für jedes $u\in N$ t.g. und eingelegt in M ist und ferner N zu einer auf der Zerlegung in die Teilmengen M_u definierten Faktormenge isomorph ist. *O. Borůvka* (Brno)

5585:

Baer, Robert M. On closure operators. Arch. Math. **10** (1959), 261–266.

In the present paper the author studies the set $\mathscr{C}(X)$ of closure operators of a partially ordered set X. First he introduces the notion of partial ordinal in X. For every $x\in X$, let $\check{x}=\{y\in X: y\geq x\}$. A non-empty subset A of X is a partial ordinal if for every $x\in X$, $\check{x}\cap A$ is non-empty and has a least element. The set $\mathscr{P}(X)$ of partial ordinals can be ordered by set inclusion and the author proves that $\mathscr{C}(X)$ and $\mathscr{P}(X)$ are dually isomorphic [cf. Ward, Ann. of Math. (2) **43** (1942), 191–196; MR **3**, 261]. It is also shown that $\mathscr{C}(X)$ is a complete lattice if and only if X contains the least upper bound of every non-empty subset. Finally the author proves that if X is a lattice then $\mathscr{P}(X)$ is closed under finite union and intersection if X is a chain which contains the least upper bound of every non-empty subset [cf. Dwinger, Indag. Math. **16** (1954), 560–563; MR **16**, 668]. *Ph. Dwinger* (Lafayette, Ind.)

5586:

Mrówka, S. Two remarks on my paper: "On the ideals' extension theorem and its equivalence to the axiom of choice". Fund. Math. **46** (1959), 165–166.

The author points out that his proof in a previous paper [Fund. Math. **43** (1956), 46–49; MR **18**, 10] that a certain statement (T_0) implies the axiom of choice is faulty. The statement was: (T_0) In a Boolean algebra an ideal disjoint from a set A is always contained in a maximal ideal disjoint from A. His proof actually showed that the following statement (T_1) implies the axiom of choice: (T_1) In a distributive lattice with maximal element I, an ideal disjoint from a non-empty set A is always contained in a maximal ideal disjoint from A.

He also shows that the following statement (T_2) implies the axiom of choice: (T_2) In a lattice with maximal element I, every proper ideal is contained in a maximal proper ideal. He states, with indication of the proof, that the following statement (T_3) is equivalent to the axiom of choice: (T_3) In a lattice with O and I elements, the members of any family of proper filters (dual ideals) are contained in the corresponding members of a family of maximal proper filters. *O. Frink* (University Park, Pa.)

5587:

Bruns, Günter. Verbandstheoretische Kennzeichnung vollständiger Mengenringe. Arch. Math. **10** (1959), 109–112.

Ein Mengensystem M heißt ein vollständiger Mengenring, wenn mengentheoretische Vereinigung und mengentheoretischer Durchschnitt jedes Teilsystems von M wieder zu M gehören. Ein Intervall $\langle a, b\rangle$ eines Verbandes $(a\neq b)$ heißt ein Sprung, wenn es nur die Elemente a, b enthält. Es wird der folgende Satz bewiesen: Genau dann ist der vollständige Verband L einem vollständigen Mengenring isomorph, wenn er den beiden folgenden Bedingungen genügt: (1) in L gelten die unendlichen Distributivgesetze $a\cap\bigcup x_s=\bigcup(a\cap x_s)$, $a\cup\bigcap x_s=\bigcap(a\cup x_s)$; (2) jedes (nichttriviale) Intervall aus L enthält einen Sprung. Es wird weiter die Verbindung dieses Satzes mit dem analogen Darstellungssatz von Raney [Proc. Amer. Math. Soc. **3** (1952), 677–680; MR **15**, 389] und Balachandran [Fund. Math. **41** (1954), 38–41; MR **16**, 212] hergestellt. {Auf S. 112 ist "$\geq$" zweimal durch "$\ngeq$" zu ersetzen.} *M. Kolibiar* (Bratislava)

5588:

Amemiya, Ichiro; and Halperin, Israel. Complemented modular lattices derived from non-associative rings. Acta Sci. Math. Szeged **20** (1959), 181–201.

This paper contains a number of extensions of von Neumann's construction of a relatively complemented modular lattice from an associative regular ring. A principal theorem is the following. Let R be an idempotent-associative, semi-regular ring. Then the lattice L of left ideals with single idempotent generators is a relatively complemented lattice with null element. If R has a right unit, then L is complemented and if R is regular, L is modular.

The authors also determine conditions on the ring R in order that the lattice of principal idempotent generated left ideals of a suitable matrix ring over R be a relatively complemented modular lattice having homogeneous bases of orders two and three respectively. Finally, conditions are given which insure that the authors' previous coordinatization construction when applied to the ideal lattice will produce a ring isomorphic to the original ring. *R. P. Dilworth* (Pasadena, Calif.)

5589:

Takeuchi, Kensuke. On free modular lattices. II. Tôhoku Math. J. (2) **11** (1959), 1–12.

The paper is a continuation of the author's paper, Japan. J. Math. **21** (1951), 53–65 [MR **14**, 529], and deals with the word problem for free modular and distributive lattices. The author gives the Hasse diagram of the well-known free modular lattice generated by $2+1+1$. This diagram solves the word problem for this lattice. In the following section the author proves that any two distinct canonical words of rank $\leqq 2$ define distinct elements of the free modular lattice. For a free distributive lattice $\mathfrak{L}$ the author gives a necessary and sufficient condition for $a \leqq b$ $(a, b \in \mathfrak{L})$, which is a modification of the known condition of Whitman for free lattices. The word problem for abstract algebras is formulated in the Appendix.

V. Vilhelm (Prague)

5590:

Crawley, Peter. The isomorphism theorem in compactly generated lattices. Bull. Amer. Math. Soc. **65** (1959), 377–379.

It is well known that in a modular lattice L the isomorphism theorem holds: if $a, b \in L$, then the quotient sublattices (intervals) $a \cup b/a$, $b/a \cap b$ are isomorphic. In this paper it is shown that the converse of this theorem holds if we replace the elements a, b by ideals A, B in L. More precisely, an arbitrary lattice is modular if and only if for any two ideals A, B of L the quotients $A \cup B/A$, $B/A \cap B$ are (as sublattices of the lattice of all ideals of L) isomorphic. This theorem is an easy consequence of the following main result of the paper. Let L be a complete lattice. An element $c \in L$ is said to be compact if for every subset $S \subseteq L$ with $c \leqq \bigcup S$ there exists a finite set $S' \subseteq S$ such that $c \leqq \bigcup S'$. L is called compactly generated if for every $a \in L$ there exists a set $C \subseteq L$ of compact elements such that $a = \bigcup C$. The main theorem then says: if for all elements a, b of a compactly generated lattice L the quotients $a \cup b/a$, $b/a \cap b$ are isomorphic, then L is modular.

V. Vilhelm (Prague)

5591:

Pierce, R. S. A generalization of atomic Boolean algebras. Pacific J. Math. **9** (1959), 175–182.

Let α be an infinite cardinal number. A Boolean algebra B is α-atomic if B contains a subset A such that (i) $0 \notin A$; (ii) every non-zero element of B contains an element of A; (iii) every directed (down) subset of elements of A of cardinality at most α has a lower bound in A. If B is α-atomic for all α, it is called ∞-atomic. The author shows that a Boolean algebra is atomic in the ordinary sense if and only if it is ∞-atomic; that the free α-representable algebra on any number of generators is α-atomic; and that any α-atomic algebra is isomorphic to a dense subalgebra of the normal completion of an α-atomic α-complete field of sets. The notions are also related to the infinite distributive laws.

Dana Scott (Chicago, Ill.)

5592:

Pierce, R. S. Representation theorems for certain Boolean algebras. Proc. Amer. Math. Soc. **10** (1959), 42–50.

Using a topological approach via the Stone space, the author reproves and generalizes results about α-complete Boolean algebras and their connection with α-complete fields of sets. The method is related to that used by R. Sikorski [Fund. Math. **36** (1949), 245–247, MR **12**, 76] and employs a suitable extension of the notion of a nowhere-dense set which is appropriate for the Stone spaces of α-complete algebras. However, the main results do not always require completeness assumptions. An interesting corollary of the technique generalizing the theorem on the representation of σ-complete algebras shows that every Boolean algebra is isomorphic to a subalgebra of the quotient of the complete Boolean algebra of all subsets of its Stone space modulo the σ-ideal of the first-category subsets; in this subalgebra every infinite sum which exists is equal to the corresponding sum in the sense of the whole quotient algebra. In this style necessary and sufficient conditions for an algebra to be α-representable are given. Some unsolved problems are formulated.

Dana Scott (Chicago, Ill.)

5593:

Dwinger, Ph. A note on the completeness of factor algebras of α-complete Boolean algebras. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 376–383.

In two previous papers [same Proc. **61** (1958), 448–456; **62** (1959), 26–35; MR **20** #6380] the author investigated the completeness of factor algebras of complete Boolean algebras. In this paper, the investigation is continued for α-complete Boolean algebras. The main result shows that if an α-complete Boolean algebra, α a transfinite cardinal, has a disjointed subset consisting of α elements, then the Boolean algebra has a factor algebra which is β-complete for every $\beta < \alpha$, but which is not α-complete. Some of the results are applied to obtain topological theorems about the associated Stone space and non-existence of measures in certain Boolean algebras.

J. Hartmanis (Schenactady, N.Y.)

GENERAL MATHEMATICAL SYSTEMS

See 5589, 5647.

THEORY OF NUMBERS

See also 5582, 5705, 5748, 5751.

5594:

Moser, L. On the minimal overlap problem of Erdös. Acta Arith. **5** (1959), 117–119.

The author gives an ingenious and elementary proof of the following theorem. Let $1 \leqq a_1 < a_2 < \cdots < a_n \leqq 2n$ be an arbitrary set of integers and by $b_1 < b_2 < \cdots < b_n \leqq 2n$ we denote the positive integers $\leqq 2n$ which are not a's. Denote by M_k the number of solutions of $a_i - b_j = k$ and put $M(n) = \max_k M_k$. The author proves that (1) $M(n) > \frac{1}{4}(2)^{1/2}(n-1)$. An example of Selfridge, Motzkin and Ralston shows that $M(n)$ can be $\leqq 0.4n$. Thus (1) is not far from being best possible. [Previously the best result was due to Świerczkowski, Colloq. Math. **5** (1958), 185–197; MR **21** #1955.]

P. Erdös (Adelaide)

5595:

Varnavides, P. On certain sets of positive density. J. London Math. Soc. **34** (1959), 358–360.

Let $0<a_1<a_2<\cdots<a_m<x$, $m>\delta x$ be an arbitrary sequence of integers. K. F. Roth proved [same J. **28** (1953), 104–109; MR **14**, 536] that if $x>x_0(\delta)$ the equation $a_i+a_j=2a_r$ is always solvable, and the author proved in a previous paper [same J. **30** (1955), 325–326; MR **17**, 946] that the number of solutions is greater than $C(\delta)x\log x$. He now shows that the number of solutions in question is greater than $C(\delta)x^2$, which is best possible except for the value of $C(\delta)$. *P. Erdös* (Adelaide)

5596:

Makowski, A. Remark on a paper of Erdös and Turán. J. London Math. Soc. **34** (1959), 480.

Let $r_3(n)$ denote the greatest integer m for which there is an increasing sequence

$$a_1<a_2<\cdots<a_m\leqq n\quad (a_i \text{ positive integers})$$

containing no three terms which are in arithmetic progression. Erdös and Turán [J. London Math. Soc. **11** (1936), 261–264] stated that $r_3(20)=8$.

The author disproves this, since in the sequence

$$1,\ 2,\ 6,\ 7,\ 9,\ 14,\ 15,\ 18,\ 20$$

no three terms are in arithmetic progression. The author states he has proved—what his counter-example suggests—$r_3(20)=9$. *S. Chowla* (Boulder, Colo.)

5597:

Schinzel, A. Sur l'équation $\phi(x+k)=\phi(x)$. Acta Arith. **4** (1958), 181–184.

Denote by $\phi(m)$ the Euler ϕ function. W. Sierpiński [Publ. Math. Debrecen **4** (1956), 184–185; MR **18**, 17] proved that for every integer k there exists an integer X such that $\phi(X+k)=\phi(X)$. In this paper this result is generalized to prove that there exists an integer a such that $\phi(X+k^a)=\phi(X)$ has two distinct integral solutions X. *N. C. Ankeny* (Cambridge, Mass.)

5598:

Negoescu, N. C. Sur la fonction $\sigma_k(n)$. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 149–152 (1959). (Romanian. French and Russian summaries)

Set as usual $\sigma_k(n)=\sum_{d|n}d^k$. Then the author shows that

$$\sigma_k(n)=\sum_{d|n}d^{k-1}\varphi(d)\sigma_{k-1}(n/d).$$

The proof uses the identities (valid for $s>\max(1,k+1)$) $\zeta(s)\zeta(s-k)=\sum_{n=1}^{\infty}\sigma_k(n)n^{-s}$ (assumed known) and $\zeta(s-k)/\zeta(s-k+1)=\sum_{n=1}^{\infty}\varphi(n)n^{-s+k-1}$ (proven in the paper). *E. Grosswald* (Princeton, N.J.)

5599:

McCarthy, P. J. On an arithmetic function. Monatsh. Math. **63** (1959), 228–230.

Let k be a positive integer, and for any arithmetic function $g(m)$ let

$$F_k(m)=\sum_{d^k|m}\mu(d)g(m/d^k).$$

The author obtains a necessary and sufficient condition on $g(m)$ in order that $F_k(m)\equiv 0 \pmod m$ shall hold for all m, thereby generalizing a result of Carlitz [Bull. Amer. Math. Soc. **43** (1937), 271–276; Amer. Math. Monthly **59** (1952), 386–387; MR **14**, 22]. *H. Davenport* (Cambridge, England)

5600:

Kanold, Hans-Joachim. Über zahlentheoretische Funktionen. III. Math. Ann. **135** (1958), 251–256.

[For parts I, II, see J. Reine Angew. Math. **195** (1955), 180–191; Math. Ann. **134** (1957), 41–46; MR **17**, 827; **19**, 839.] For a mapping f of the set of positive integers into itself let $\mathfrak{N}_s$ be the set of positive integers n such that s distinct numbers $n_1, n_2, \cdots, n_s$, different from n, exist for which $f(n_1)=f(n_2)=\cdots=f(n_s)$, and let $\mathfrak{N}_s'$ be the corresponding set if n runs through a subset $\mathfrak{N}'$ of the set of positive integers. The author gives an example of a function with $\lim\inf f(n)=0$ and for which $\mathfrak{N}_s$ is the empty set for $s\geqq 2r-1$, where r is a given integer >1. This example shows that theorem 4 in part I does not remain true if lim is replaced by lim inf. He further notes that theorem 5 in part I can be formulated as follows. Let $\mathfrak{N}'$ be a subset of the set of positive integers and let $\mathfrak{F}'$ be the set of all $f(n')$, $n'\in\mathfrak{N}'$. Suppose that the asymptotic density of $\mathfrak{F}'$ is zero and that $\sum_{n'\leq N}f(n')/n'=O(N)$ (summation over $n'\in\mathfrak{N}'$ only). Then for the upper and lower densities we have $\bar{D}(\mathfrak{N}_s')=\bar{D}(\mathfrak{N}')$, $\underline{D}(\mathfrak{N}_s')=\underline{D}(\mathfrak{N}')$. He finally proves: If $f(n)$ is multiplicative $(f(n_1n_2)=f(n_1)f(n_2)$ for $(n_1,n_2)=1)$ such that $\lim\inf f(n)=\infty$, $\sum_{n\leq x}f(n)/n=O(x)$ and the set $\mathfrak{F}$ of values $f(n)$ has density zero, then the set $\mathfrak{F}$ contains an infinite number of squares of integers. *H. D. Kloosterman* (Leiden)

5601:

Salié, Hans. Zum Wertevorrat der Dedekindschen Summen. Math. Z. **72** (1959/60), 61–75.

For real x let $((x))=0$ if x is an integer and $x-[x]-\frac{1}{2}$ otherwise. Let

$$s(m,n)=\sum_{v \bmod n}\left(\left(\frac{v}{n}\right)\right)\left(\left(\frac{mv}{n}\right)\right),\quad (m,n)=1,\quad n>0,$$

denote the Dedekind sum and put $D(m,n)=12ns(m,n)$. The author introduces the function $T(m,n)$ uniquely defined for all natural integers m, n by means of

$$T(m,n)+T(n,m)=[m/n]+[n/m]-2,\quad m>0,\quad n>0,$$
$$T(m',n)=T(m,n),\quad m'\equiv m \pmod n,$$

and first proves with the aid of the Euclidean algorithm that

$$D(m,n)=m+\bar{m}-n+nT(m,n),\quad m\geqq 1,\quad n>1,$$

where $\bar{m}$ is the smallest positive solution of the congruence $m\bar{m}\equiv 1 \pmod n$, $(m,n)=1$. He then systematically develops a body of arithmetical theorems concerning the values assumed by $T(m,n)$. A typical theorem of this sort states that if $x\geqq 1$, then $T(amx\pm 1,\ m^2x)$ equals $\pm((a,m)^2x-3)$ or $\pm(x-2+[1/x])$ according as $m\geqq 2$ or $m=1$ respectively. Corresponding results for $D(m,n)$ are also derived. Particularly noteworthy is the transformation formula: if $(m,n)=1$, then for every integer $x\geqq 0$

$$D(nx+m,\ ((nx+m)^2-D(m,n)(nx+m)+1)/n)=D(m,n).$$

One application of these results is that the values of $D(m,n)$ are restricted to the five residue classes 0, ± 2,

$\pm 6 \pmod{18}$. In each of these classes there are infinitely many values of $D(m, n)$.

The present paper is closely related to two earlier papers by Rademacher [Math. Z. **63** (1956), 445–463; MR **18**, 114] and Dieter [Abh. Math. Sem. Univ. Hamburg **21** (1957), 109–125; MR **19**, 395]. *A. L. Whiteman* (Princeton, N.J.)

5602:

Carlitz, L. Note on a formula of Hermite. Math. Mag. **33** (1959/60), 7–11.

The Staudt-Clausen theorem for Bernoulli numbers implies the congruence

$$\sum_{0<r(p-1)<n} \binom{n}{r(p-1)} \equiv 0 \pmod p,$$

where p is an odd prime. If $p=2m+1$, let

$$S = \sum_{0<rm<n} (-1)^{n-rm} \binom{n}{rm}.$$

The author proves that $S \equiv 0 \pmod p$ unless $p-1 | n+k$, $0 \leq k \leq m$, in which case $S \equiv \binom{m}{k} \pmod p$. He also relates the congruences for S to a result analogous to the Staudt-Clausen theorem. *T. M. Apostol* (Pasadena, Calif.)

5603:

Jarden, Dov. Consecutive integers divisible by many primes. Riveon Lematematika **13** (1959), 25. (Hebrew. English summary)

"Theorem: For any two given positive integers n, ν there exist n consecutive positive integers each of which is divisible by at least ν different primes."

Author's summary

5604:

Izbicki, Herbert. Über eine Klasse zusammengesetzter Zahlen. Monatsh. Math. **63** (1959), 309–316.

Let $b-1$, m, and $n-1$ be positive integers. The number $N=\sum_{k=0}^{n-1} b^{mk}$ is then composite whenever n is composite, since an explicit factorization is at hand in this case. The author exhibits another explicit factorization of N in case $m>1$ and $(m, n)=1$; so N can be prime only in case n is prime and m is either 1 or is divisible by n.

A. Sklar (Chicago, Ill.)

5605:

Schinzel, André. Sur quelques propositions fausses de P. Fermat. C. R. Acad. Sci. Paris **249** (1959), 1604–1605.

Fermat stated that (p denotes a typical prime): (1) If $p|(3^x+1)$, then $p \not\equiv \pm 1 \pmod{12}$; (2) If $p|(5^x+1)$, then $p \not\equiv \pm 1 \pmod{10}$. Sierpiński remarked that proposition (1) is false when $p \equiv 1 \pmod{12}$ since $61|(3^5+1)$. He suggested that there are infinitely many primes $p \equiv 1 \pmod{12}$ dividing numbers of the form 3^x+1. This suggestion is proved by the author, quoting a theorem of Birkhoff and Vandiver [Ann. of Math. (2) **5** (1904), 173–180]. The author's second theorem disproves (2) for infinitely many primes $p \equiv 1 \pmod{10}$, again quoting Birkhoff and Vandiver. Finally he disproves proposition (2) for infinitely many primes $p \equiv -1 \pmod{10}$, but this needs the law of biquadratic reciprocity and also the theorem that there are infinitely many primes representable simultaneously by the forms $20k+9$ and $25x^2+4y^2$, which is included in a well-known theorem of A. Meyer.

S. Chowla (Boulder, Colo.)

5606:

Subba Rao, K. Some properties of arithmetic progressions. Amer. Math. Monthly **66** (1959), 582–584.

By means of identities and the well-known procedure of making a^2+kd^2 a perfect nth power, the author proves that the diophantine equation $\sum_{i=1}^{m} x_i^2 = mz^n$, where $x_1, x_2, \cdots, x_m$ are integers in arithmetic progression and z, n are integers, has infinitely many solutions. The same result is obtained for the equations $\sum_{i=1}^{m} x_i^s = z^n$ in the cases $s=2$, n odd and $s=3$, $n \not\equiv 0 \pmod 3$.

W. Ljunggren (Oslo)

5607:

★Gelfond, A. O. The solution of equations in integers. Popular Lectures on Mathematics. Translated from the 2nd Russian edition by Leo F. Boron. P. Noordhoff, Ltd., Groningen, 1960. 72 pp. Paperbound.

The Russian original was listed as MR **20** #6387. The foreword to this American edition is a reproduction of a letter from the author; it remarks, in part: "This booklet is accessible to the more advanced high school students, ..., to teachers of mathematics, and to engineers."

5608:

Petersson, Hans. Über Darstellungsanzahlen von Primzahlen durch Quadratsummen. Math. Z. **71** (1959), 289–307.

This paper continues the study of a problem begun by the author in an earlier paper [Math. Nachr. **14** (1956), 361–375; MR **18**, 867]. In that paper the author factors the polynomial whose roots are the primitive qth roots of unity (q a prime >3) and shows how to determine the coefficients of the factors in terms of coefficients of the Eisenstein series expansion of certain modular functions. This coefficient is the leading term of $a_k(q)$ which is the number of representations of q as the sum of k squares. A detailed study of the modular functions concerned reveals various relations between these numbers, especially for $k \leq 11$. In particular certain congruence relations are obtained for numbers designated by e_k^*; these differ from $e_k(q)$ by trivial factors (e.g., $e_{11}(q)=3960e_{11}^*$ if $q \equiv 7 \pmod 8$). For example, if $q \equiv 7 \pmod 8$, $q>7$, then $e_7^* \equiv e_{11}^* \equiv 0 \pmod 4$, $e_7^* \equiv e_{11}^* \pmod 8$, $e_{11}^* \equiv 2(q+1)h \pmod 3$, $e_{11}^* \equiv 2(q^2-1)h \pmod 5$; here h is the class number of the quadratic field determined by $\sqrt{-q}$.

H. W. Brinkmann (Swarthmore, Pa.)

5609:

Makowski, A. Sur quelques problèmes concernant les sommes de quatre cubes. Acta Arith. **5** (1959), 121–123.

A conjecture of W. Sierpiński, that every integer g has infinitely many representations in the form $g=x^3+y^3-z^3-t^3$ (x, y, z, t integers >0), is verified here for the range $301 \leq |g| \leq 350$, using the method developed by A. Schinzel and W. Sierpiński [Acta Arith. **4** (1958), 20–30; MR **20** #273], where the verification is given for $|g| \leq 300$, omitting the values $|g|=148$, 257, 284. The author corrects some typographical errors in the tables in the paper just cited and gives an affirmative answer to a question raised there,

whether the equation, $x^3-y^3-z^3-t^3=2$, has infinitely many solutions in positive integers. He also extends a result of Chao Ko [J. London Math. Soc. **11** (1936), 218–219], that every integer g, $|g|\leq 100$, has a representation in the form $g=x^3+y^3+2z^3$, x, y, z integers, except possibly for $|g|=76$, 99. The extension is to the range $101\leq|g|\leq 220$ with the values $|g|=113$, 148, 183, 190, 195 remaining undecided. *R. J. Levit* (San Francisco, Calif.)

5610:

Korobov, N. M. Partially rational trigonometric sums. Dokl. Akad. Nauk SSSR **125** (1959), 1193–1195. (Russian)

Let $f(x)=\alpha_1 x+\cdots+\alpha_{n+1}x^{n+1}$ be a polynomial such that α_i is real for $1\leq i\leq s$ and $q\alpha_i$ is an integer for $s<i\leq n+1$. Here q is a prime and $(q, q\alpha_{n+1})=1$. By using estimations due to Weyl, Vinogradov and himself [see, in particular, Korobov, Uspehi Mat. Nauk **13** (1958), no. 4 (82), 185–192; MR **21** #4939], the author proves that

$$\left|\sum_{P<x\leq P+q} e^{2\pi i f(x)}\right| \leq q^{1-\gamma/(s^2\log 2s)}\exp\left\{\beta\left(\frac{\log n}{s^2\log 2s}+s\log^2 2s\right)\right\},$$

where β and γ are positive absolute constants, and P is an arbitrary integer. Various improvements are mentioned. *R. A. Rankin* (Glasgow)

5611:

Vinogradov, I. M. Estimate of a prime-number trigonometric sum. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 157–164. (Russian)

This paper contains an estimate for sums of the type

$$S=\sum_{p\leq P}\exp(2\pi i A p^\alpha),$$

where p runs through primes. Let α be a constant ≥ 6, and suppose that α differs from the nearest integer by at least $3^{-\alpha}$. Suppose that $1\leq A\leq P^\alpha$. Then $|S|\ll P^{1-\rho}$, where $\rho=(34{,}000{,}000\alpha^2)^{-1}$. The proof is complicated but highly ingenious, and involves a combination of ideas from the author's recent work on trigonometric sums [Izv. Akad. Nauk SSSR. Ser. Mat. **22** (1958), 161–164, 577–584; MR **21** #2624; **20** #5264] and from his earlier work on sums extended over primes [Chapter IX of Trudy Mat. Inst. Steklov **23** (1947); English transl. by Roth and Davenport, Interscience, New York, 1954; MR **10**, 599; **15**, 941]. The condition that α is not near an integer is used to ensure that the coefficients in the Taylor expansion of $A(x_0+x)^\alpha$ are not too small. *H. Davenport* (Cambridge, England)

5612:

Mitrović, Dragiša. Le théorème Ω relatif aux dérivées de la fonction ζ de Riemann. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 13–18. (Serbo-Croatian summary)

It is proved, for example, that if $\sigma>1$, $k=0, 1, 2, \cdots$,

$$|\zeta^{(k)}(\sigma+it)|\geq(1-\varepsilon)\zeta^{(k)}(\sigma)$$

for some arbitrarily large values of t. The results seem to be immediate consequences of well-known ideas due to H. Bohr. *E. C. Titchmarsh* (Oxford)

5613:

Lehmer, D. H. On the exact number of primes less than a given limit. Illinois J. Math. **3** (1959), 381–388.

This paper contains the values of $\pi(x)$ for certain large numbers x, such as 10^{10}. The largest number x for which $\pi(x)$ had previously been calculated was 10^9 [E. D. F. Meissel, Math. Ann. **25** (1885), 251–257]. The author has found mistakes in Meissel's calculations, and corrects them. He refines the formula on which Meissel's calculations were based, so as to make it suitable for use in high-speed computing machines. He also gives a new proof of Meissel's formula, and indicates the applicability of his method to other problems, such as finding the sum of the kth powers of the primes less than x. *T. Estermann* (London)

5614:

Rieger, G. J. Zur Wienerschen Methode in der Zahlentheorie. Arch. Math. **10** (1959), 257–260.

The author refers to a paper by M. Glatfeld in which the prime number theorem for arithmetic progressions is proved by Wiener's method [J. London Math. Soc. **32** (1957), 67–73; MR **18**, 874], and announces his intention of proving by Wiener's method the prime number theorem for ideal classes $\mathfrak{H}$ mod $\mathfrak{f}$ of an algebraic number field K of degree n. However, the two authors use different aspects of Wiener's method. Glatfeld used Wiener's original theory (as modified by H. R. Pitt) in which the main condition is the non-vanishing of a Fourier transform. The present author uses the Wiener-Ikehara theorem in which the main condition is of the form $f(s)=A(s-1)^{-1}+g(s)$ $(s=\sigma+it)$ where $f(s)$ is the appropriate generating function and $g(s)$ is continuous for $\sigma\geq 1$. The main part of the paper is devoted to a study of the properties of $f(s)$, and the final conclusion is reached by a routine application of the Wiener-Ikehara theorem.

The author remarks that the proof is elementary in the sense that it does not involve the residue theorem of complex function theory. It should be pointed out, however, that such proofs have existed in the literature (at least implicitly) for half a century. See, for example, E. Landau, Prace Mat.-Fiz. **21** (1910), 97–177; Satz XVIII; the application of Cauchy's theorem on p. 175 can be avoided by taking the integral along the line $\sigma=1+\delta$ instead of $\sigma=2$ and making $\delta\to 0+$ (under the integral sign by continuity and uniform convergence), and the formula in the footnote on p. 174 (with the integral taken along $\sigma=1+\delta$ instead of $\sigma=2$) can, of course, be proved without Cauchy's theorem. The significant contribution of Wiener's methods to the logical scheme is that they avoid all reference to the order of magnitude of $f(s)$ for large $|t|$. *A. E. Ingham* (Cambridge, England)

5615:

Wang, Yuan. On sieve methods and some of their applications. Acta Math. Sinica **9** (1959), 87–100. (Chinese. English summary)

The author gives detailed proofs of the following results. For $k>5$ denote by w the smallest integer satisfying

$$w+1\geq\frac{5.64527}{4.8396}+\frac{3.65}{4.8396}\log\frac{5k-w}{w+5},$$

and put $n=k+1$ if $1\leq k\leq 5$, but $n=k+w$ if $k>5$. (1) If $F(x)$ is an irreducible integral-valued polynomial of degree

k without fixed prime divisor, there are infinitely many integers x such that $F(x)$ is the product of at most n primes. (2) For sufficiently large x at least one integer between x and $x+x^{1/k}$ has at most n prime factors. (3) For sufficiently large x at least one integer between x and $x+x^{10/17}$ has at most two prime factors; at least one between x and $x+x^{20/49}$ has at most three prime factors; and at least one between x and $x+x^{9/902}$ has at most 100 prime factors. *K. Mahler* (Manchester)

5616:

Fröhlich, A. The restricted biquadratic residue symbol. Proc. London Math. Soc. (3) **9** (1959), 189–207.

The author states some of the principal properties of the quadratic residue symbol of elementary number theory, including the existence of a law of reciprocity. For $m>2$, it is impossible to define a rational mth power residue symbol to which any of the stated properties extend, but the author defines what he calls a restricted biquadratic residue symbol, $\left[\frac{x}{y}\right]$, which has analogous properties to those of the quadratic residue symbol. His definition is slightly more general than an earlier one of L. Redei [cf. Monatsh. Math. Phys. **48** (1939), 43–60; MR **1**, 134]. He gives a characterization of his symbol in terms of congruence class groups, derives analogues to Euler's and Gauss's criteria, and shows that the symbol has the following multiplicative property. If $\left[\frac{y}{t_1}\right]$ and $\left[\frac{y}{t_2}\right]$ are both defined then $\left[\frac{y}{t_1t_2}\right]$ is defined and

$$\left[\frac{y}{t_1t_2}\right]=\left[\frac{y}{t_1}\right]\left[\frac{y}{t_2}\right].$$

R. Hull (Santa Monica, Calif.)

5617:

Siegel, Carl Ludwig. Zur Bestimmung des Volumens des Fundamentalbereichs der unimodularen Gruppe. Math. Ann. **137** (1959), 427–432.

In an earlier work [Trans. Amer. Math. Soc. **39** (1936), 209–218] the author verified Minkowski's formula for the volume of the fundamental region of the unimodular group of the nth order, by analytical methods involving the zeta-function for the ring of $n\times n$ matrices over the rational field. This was generalized to the case of an arbitrary algebraic number field. The present note draws attention to a gap in his proof (p. 211, where two divergent integrals are introduced) and remedies it.

J. H. H. Chalk (Hamilton, Ont.)

5618:

Wohlfahrt, Klaus. Über Operatoren Heckescher Art bei Modulformen reeller Dimension. Math. Nachr. **16** (1957), 233–256.

The author generalizes the theory of Hecke operators to automorphic forms of non-integral dimensions. If f is a modular form of class $K=\{\mathrm{K}, -r, \kappa\}$ (i.e., belongs to the group K, has dimension $-r$ and multiplier system κ) he considers linear combinations $g=\sum_Q c_Q\cdot f|Q$, where Q runs through a finite set of matrices $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with integral a, b, c, d such that $ad-bc$ is a fixed positive integer n and where $f(\tau)|Q=(c\tau+d)^{-r}f(Q\tau)$. He tries to choose the coefficients c_Q in such a way that $f\to g$ is a linear mapping of K into the class $\Lambda=\{\Lambda, -r, \lambda\}$.

This leads to the consideration of sums

$$g=\sum(\lambda(V))^{-1}f|Q|V,$$

where V runs through a certain finite set of matrices of Λ (in fact a complete set of representatives of the left cosets of $\Delta=\Lambda\cap Q^{-1}\mathrm{K}Q$ in Λ). It must be assumed that Δ has finite index in Λ and that on Δ the multiplier λ is identical with the multiplier (denoted by $\kappa|Q$) which corresponds on $Q^{-1}\mathrm{K}Q$ with the multiplier κ on K. The mapping $f\to g$ is then denoted by $T_K{}^{\Lambda}\langle Q\rangle$. The definition of this operator applies even to certain subgroups of the group of all real unimodular (2×2)-matrices. In the case of the modular group and if r is an even number $\geqq 4$ the relation with Hecke's operators is easily established by means of a normalized representation of $T_K{}^{\Lambda}\langle Q\rangle$ as a linear combination of operators in which the matrices Q are of type $\begin{pmatrix}a & bN\\ 0 & d\end{pmatrix}$, where $ad=n$, $a>0$, $b \bmod d$, $(a, b, d)=1$. This normalized representation applies to the case in which $\mathrm{K}\cap\Lambda$ contains the principal congruence subgroup $\Gamma(N)$ of level N (and some other conditions are satisfied). It leads also to the relation between the Fourier coefficients in the expansion (at ∞) for f and those in the expansion for $f|T_K{}^{\Lambda}\langle Q\rangle$. The author considers an application of the general theory to the theory of quadratic forms in an odd number of variables. The theta-series corresponding to these quadratic forms (which are modular forms of a dimension $\equiv\frac{1}{2}\pmod 1$) or certain linear combinations of them can be represented as linear combinations of modular forms of the same class, whose Fourier coefficients satisfy relations of type

$$c(m)c(n)=\sum_{t|(m,n)}\gamma(t)c(mn/t^2)$$

if m and n are odd squares and where $\gamma(t)$ is a certain simple arithmetical function involving quadratic residue symbols. The special case $\vartheta^3(\tau)\eta(\tau)$, where $\vartheta(\tau)=\sum_{n=-\infty}^{+\infty}\exp\pi i\tau n^2$ and $\eta(\tau)$ is Dedekind's function, leads to relations between class numbers of imaginary quadratic number fields as a consequence of the fact that these class numbers can be expressed by means of the number of representations of positive numbers as a sum of three squares. *H. D. Kloosterman* (Leiden)

5619:

Fel'dman, N. I. On the measure of transcendence of the number π and of the logarithms of algebraic numbers. Dokl. Akad. Nauk SSSR **126** (1959), 1214–1215. (Russian)

The author states that the method of his paper [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 53–74; MR **12**, 595] can be improved so as to give the following results. Let $\zeta=\log\alpha$ be the natural logarithm of an algebraic number $\neq 0$, $\neq 1$. There exists a positive constant γ_0 depending only on α, such that, if $H>\exp(n^4)$,

$$|a_0+a_1\zeta+\cdots+a_n\zeta^n|>H^{-\gamma_0n^2(\log(n+1))^2}$$

for all integers $a_0, a_1, \cdots, a_n$ not all zero satisfying $\max_{0\leq h\leq n}|a_h|\leq H$. For $\zeta=\pi$ the better estimate

$$|a_0+a_1\pi+\cdots+a_n\pi^n|>H^{-\gamma n\log(n+1)}$$

if $H > \exp(n^4)$ holds where γ is an absolute constant. Both estimates are much sharper than the so far best due to the reviewer [Philos. Trans. Roy. Soc. London, Ser. A **245** (1953), 371–398; Nederl Akad. Wetensch. Proc. Ser. A **56** (1953), 30–42; MR **14**, 624, 957].

K. Mahler (Manchester)

5620:

★Schneider, Théodor. Introduction aux nombres transcendants. Traduit de l'allemand par P. Eymard. Gauthier-Villars, Paris, 1959. viii+151 pp. Paperbound: 3500 francs; $7.30.

The original [*Einführung in die transzendenten Zahlen*, Springer, Berlin-Göttingen-Heidelberg, 1957] is reviewed in MR **19**, 252.

5621:

Sharma, A. On Newton's method of approximation. Ann. Polon. Math. **6** (1959), 295–300.

The results of J. Mikusiński [Ann. Polon. Math. **1** (1954), 184–194; MR **15**, 954] are here extended to a class of numbers $\sqrt{C}+L$, which have a continued fraction expansion of the form (A) $a(a_1, a_2, \cdots, a_p)$ of period p, as follows: (I) If x_n is the $(p-1)$th approximant of a from (A) (p being the number of terms in a period not necessarily primitive), then the number x_{n+1} given by (B) $x_{n+1} = (x_n^2 - B)/(2x_n + A)$ is equal to the $(2p-1)$th approximant of a, where $a(=\sqrt{C}+L)$ is a root of the equation $x^2+Ax+B=0$. (II) If the number a has a period of length $2i$ and is given by $a = a(a_1, a_2, \cdots, a_{i-1}, b, a_{i-1}, \cdots, a_1, d)$, so that the period is symmetric except for the last term, then all iterations given by Newton's formula on taking $x_n = (i-1)$th approximant of a are also approximants of a.

E. Frank (Chicago, Ill.)

5622:

Barnes, E. S. Criteria for extreme forms. J. Austral. Math. Soc. **1** (1959/61), part 1, 17–20.

The positive definite quadratic form $f(x) = \sum a_{ij}x_ix_j$ $(a_{ij} = a_{ji})$ of determinant $\|a_{ij}\| = D$ and minimum M for integral $x \neq 0$ is said to be extreme if the ratio $\gamma_n(f) = M/D^{1/n}$ is a local maximum for small variations in the coefficients; it is absolutely extreme if $\gamma_n(f) = \gamma_n$ where $\gamma_n = \max \gamma_n(f)$ (taken over all forms in n variables). The minimal vectors of f are the integral solutions $x = \pm m_1, \pm m_2, \cdots, \pm m_s$ of $f(x) = M$. Let H be any subset of the minimal vectors, say $m_1, m_2, \cdots, m_t$ $(t \leq s)$. The author calls f H-perfect if f is uniquely determined by H and its minimum M; he calls f H-eutactic if the adjoint $F(x) = \sum A_{ij}x_ix_j$ is expressible as

$$F(x) \equiv \sum \rho_k(m_k'x)^2$$

with $\rho_k > 0$ $(k = 1, 2, \cdots, t)$. These definitions reduce to the accepted definitions of the terms perfect and eutactic if H is the set of all minimal vectors. Finally, let G denote the group of automorphs of f.

It is well known that f is extreme if and only if f is perfect and eutactic. The author proves, in a very simple way, the following two criteria for extreme forms. (1) f is extreme if and only if there exists a subset H of its minimal vectors such that f is H-perfect and H-eutactic. (2) If there exists a subset H of the minimal vectors of f such that f is H-perfect and G is transitive on H, then f is extreme.

These results have obvious important practical consequences in the calculations required to check that a form is extreme [see next review].

W. Moser (Winnipeg, Man.)

5623:

Barnes, E. S.; and Wall, G. E. Some extreme forms defined in terms of Abelian groups. J. Austral. Math. Soc. **1** (1959/61), part 1, 47–63.

In this paper the authors describe N-variable forms $(N = 2^n)$ which are extreme and for which $\gamma_n(f) = (N/2)^{1/2}$. It is well known [see J. F. Koksma, *Diophantische Approximationen*, Springer, Berlin, 1936, Ch. II, § 6] that $1/2\pi e \leq \liminf \gamma_n/n \leq \limsup \gamma_n/n \leq 1/\pi e$, but this is the first time that an infinite sequence of extreme forms with $\gamma_n(f)$ unbounded has been constructed.

Let V be the n-dimensional vector space over GF(2). In N-dimensional Euclidean space, consider integral vectors $x = (x_\alpha)$ with coordinates x_α indexed by the $N = 2^n$ elements of V. If W is any subset of V, $[W]$ denotes the vector x defined by $x_\alpha = 1$ if $\alpha \in W$, $x_\alpha = 0$ if $\alpha \notin W$. Let $\lambda_0, \lambda_1, \cdots, \lambda_n$ be integral exponents satisfying $\lambda_0 = 0$, $\lambda_r - 1 \leq \lambda_{r-1} \leq \lambda_r$ for $1 \leq r \leq n$. Denote by $\Lambda(\lambda) = \Lambda(\lambda_0, \lambda_1, \cdots, \lambda_n)$ the sublattice of Γ (the integer lattice) generated by all vectors $2^{\lambda_{n-r}}[C_r]$ where C_r runs through all cosets in V (considered as an additive group); r is the vector space dimension of C_r. The authors define $f_{(\lambda)}$ to be the N-dimensional form with lattice $\Lambda_{(\lambda)}$, so that the values assumed by $f_{(\lambda)}$ for integral values of its variables are those of $x^2 = \sum_{\alpha \in V} x_\alpha^2$ for $x \in \Lambda(\lambda)$. In other words, if $\Lambda(\lambda)$ is specified by $\xi = Tx$ (x integral), then $f_{(\lambda)}(x) = x'T'Tx$.

The authors prove, using the results of the paper reviewed above, that "most" of the forms $f_{(\lambda)}$ are extreme, and give criteria for determining which are extreme. A simple computation shows that the extreme forms corresponding to the exponent set $\lambda_r = [r/2]$, $0 \leq r \leq n$, have $\gamma_n(f) = (N/2)^{1/2}$. This yields a large number of extreme forms nearly all of which are new. A table at the end of the paper gives the exponent set (λ), $\log_2 M$, $\log_2 D$, s, and $\log_2 \Delta$ (where $\Delta = (2/M)^N D$) of the extreme forms $f_{(\lambda)}$ for $N = 4, 8, 16, 32$. Also, some further results which may be obtained in similar fashion are indicated, e.g., $\gamma_{15} \geq 2^{7/8}$, $\Delta_{15} \leq 2^{1/6}$.

W. Moser (Winnipeg, Man.)

FIELDS

See also 5614.

5624:

★Artin, E. Galoissche Theorie. Mathematisch-Naturwissenschaftliche Bibliothek, **28**. B. G. Teubner Verlagsgesellschaft, Leipzig, 1959. iv+86 pp. DM 5.30.

A translation by Viktor Ziegler from the English [*Galois theory*, 2nd ed., Univ. of Notre Dame, Indiana, 1944; MR **5**, 225]. In the present edition, certain changes have been made in some of the proofs of the first two parts, and the third part, entitled "Applications" and written by A. N. Milgram, has been completely revised.

5625:

Faddeev, D. K.; and Skopin, A. I. Proof of a theorem of Kawada. Dokl. Akad. Nauk SSSR **127** (1959), 529–530. (Russian)

For the number-theoretical result quoted in the title see J. Fac. Sci. Univ. Tokyo, Sect. I 7 (1954), 1–18 [MR 16, 6]. The present authors give a simpler access to the algebraic part of the theorem. The group of a normal algebraic p-extension k of a local field k_0, of degree n_0 over R_p and containing the pth roots of unity, can be represented in the form S/H, where S is a free group with n_0+2 generators. If all the fields k are contained in a fixed extension K, then there is a one-to-one correspondence between the fields k and the appropriate normal subgroups H of S. The authors show that the H that arise for various k's are precisely those that contain a certain fixed word in S.

K. A. Hirsch (London)

5626:

Feit, Walter. On p-regular extensions of local fields. Proc. Amer. Math. Soc. **10** (1959), 592–595.

Let K be a complete field with respect to a discrete valuation whose residue class field is finite and contains $q=p^f$ elements. A Galois extension over K is called a p-regular extension if its degree is prime to p. For any triple of integers a, b, c satisfying $0\leq c<a$, $0<b$, $(b,q)=1$, $c(q-1)\equiv q^b-1\equiv 0 \bmod a$, let $G(a,b,c)$ denote the group of order ab generated by two elements x, y satisfying the relations $x^a=y^bx^c=1$, $y^{-1}xy=x^q$. The author proves that a finite group G is the Galois group of a p-regular extension of K if and only if G is isomorphic to a group of the form $G(a,b,c)$ and that the number of such extensions of K with Galois group G is equal to the number of triples a, b, c such that $G(a,b,c)\simeq G$.

Let G and G' be finite groups of the type considered above, with a homomorphism ϕ of G' onto G, and let L be a p-regular extension of K with Galois group G. The author also shows by an example that for certain G, G', ϕ, and L, there is no p-regular extension of K containing L such that its Galois group over K is G' and that the given homomorphism ϕ is the natural homomorphism of Galois theory.

K. Iwasawa (Cambridge, Mass.)

ALGEBRAIC GEOMETRY

See also 5659, 5660.

5627:

Lense, Josef. Bemerkungen zur Erzeugung der kubischen Raumkurven durch drei projektive Ebenenbüschel. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1958**, 111–115.

The author justly calls attention to a gap in the discussion by B. L. van der Waerden [*Einführung in die algebraische Geometrie*, Springer, Berlin, 1939; p. 43] of the degenerate cases that can arise in the generation of a twisted cubic by three projective pencils of planes. He gives a complete analysis of the possible degenerations.

J. A. Todd (Cambridge, England)

5628:

Rionero, Mario. Curve algebriche piane e trasformazioni quadratiche. Period. Mat. (4) **37** (1959), 110–115.

Einige Beispiele von rationalen Kurven vierter und sechster Ordnung, die durch quadratische bzw. cubische Cremona-Transformationen in Kegelschnitte transformiert werden.

O.-H. Keller (Halle)

5629:

Godeaux, Lucien. Sulle superficie di ordine $2n$ con una conica multipla di ordine n. Ann. Univ. Ferrara. Sez. VII. (N.S.) 7 (1957/58), 1–7. (French summary)

The equations are obtained of a surface F^{2n} in S_3, of order n with an n-ple conic, which is transformed into itself by a cyclic homology of prime order p (a divisor of n), as well as of the surface in S_4, intersection of a general quadric with an n-ic hypersurface, of which F^{2n} is the projection. The trajectories of the group generated by this homology form an involution of order p on the surface, with n isolated united points on the multiple conic. A model of the involution is obtained as a surface of order $2np$ in S_r, where $r=\frac{1}{2}(p+1)(p+2)$. The canonical system on F^{2n} is also studied.

P. Du Val (London)

5630:

Godeaux, Lucien. Sur les surfaces algébriques de genres nuls à courbes bicanoniques irréductibles. Rend. Circ. Mat. Palermo (2) 7 (1958), 309–322.

Let F be a regular surface of genus $p_a(=p_g)\geq 2$, having a birational self transformation T of period p without fixed points, and let I be the involution in which each set consists of the p transforms of any one point under powers of T. If an image F' of I (e.g., the projective model of a linear system on F compounded with I) is regular of genus $p_a'\geq 1$, the canonical system $|K|$ on F has p linear subsystems compounded with I, one of which, of freedom $p_a'-1$, is the canonical system $|K'|$ on F', the others being of freedom p_a'. Also $p_a+1=p(p_a'+1)$.

If however $p_a'=0$, $p=p_a+1$, and $|K|$ contains $p-1$ individual curves $K_1,\cdots,K_{p-1}$ compounded with I; for a suitable ordering of the suffixes of these, their images $\Gamma_1,\cdots,\Gamma_{p-1}$ on F' have the property that

$$\Gamma_i+\Gamma_j\equiv\Gamma_k+\Gamma_l\Leftrightarrow i+j\equiv k+l \pmod p;$$

the (purely virtual) canonical system $|K'|$ is the difference $\Gamma_i+\Gamma_j-\Gamma_k$, $k\equiv i+j \pmod p$; the adjoint system of Γ_i is $|\Gamma_j+\Gamma_k|$, $j+k\equiv i \pmod p$; the bicanonical system is $|\Gamma_i+\Gamma_j|$, $i+j\equiv 0 \pmod p$, and contains, according as p is even or odd, $p/2-1$ reducible curves $\Gamma_i+\Gamma_{p-i}$ and one curve $2\Gamma_{p/2}$, or $(p-1)/2$ reducible curves $\Gamma_i+\Gamma_{p-i}$; these are linearly independent and are a base for the bicanonical system, so that $P_2=p/2$ or $(p-1)/2$; the general bicanonical curve is irreducible. The tricanonical system is $|\Gamma_i+\Gamma_j+\Gamma_k|$, $i+j+k\equiv 0 \pmod p$.

The author has already given actual examples for $p_a=4,7$ [Acad. Roy. Belg. Bull. Cl. Sci. (5) **35** (1949), 688–693; **44** (1958), 738–749; MR **11**, 393; **21** #3413].

P. Du Val (London)

5631:

Spampinato, Nicolò. Sull'invariante di Zeuthen-Segre e sulla differenza fra la classe e l'ordine di una superficie algebrica. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 208–219.

Given a surface in complex three-dimensional space, by utilising the familiar relations between order class, genus, and number of cusps and inflexions for a plane curve, applied to the general plane sections of the surface and of its general tangent cone, the author obtains a number of expressions for the Zeuthen-Segre invariant of the surface.

P. Du Val (London)

5632:

Marchionna, Ermanno. Sopra alcuni problemi della teoria algebrico-proiettiva delle varietà algebriche sghembe. Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 59–80.

An expository lecture on algebraic varieties other than hypersurfaces, especially V_d's in S_r which are non-singular and either are the complete intersection of $r-d$ hypersurfaces, or else together with another V_d', likewise non-singular and meeting V_d in a W_{d-1}, make up such a complete intersection. Results are given on the deficiency, superabundance, and specialty of the systems traced on V_d by hypersurfaces of any order, relating these to the irregularities of V_d, and to corresponding invariants of V_d'. Conditions are given for a V_d to be in fact a complete intersection, or the projection of a V_d in $S_{r'}$, say $(r'>r)$, from an $S_{r'-r-1}$ not meeting it. There are no proofs but an extensive bibliography to which detailed reference is made in the text, except that for one or two results the author refers to forthcoming papers of his own.

P. Du Val (London)

5633:

Marchionna, Ermanno. Una dimostrazione algebrico-geometrica del teorema di Riemann-Roch relativo alle varietà algebriche classiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 160–171.

In complex r-dimensional projective space S_r, V_d is an irreducible non-singular algebraic variety of $d \geqq 2$ dimensions, and $|D|$ a complete linear system of hypersurfaces of V_d (i.e., of V_{d-1}'s on V_d). The Riemann-Roch theorem is taken in the form

$$\dim|D| \equiv \delta|D| + (-1)^{d+1}j|D| + s|D|,$$

where $\dim|D|$ is the actual freedom (dimension) of $|D|$, $\delta|D|$ its virtual freedom (a function of the arithmetic genera of V_d, D, and the successive characteristic systems of $|D|$), $j|D|$ is the speciality, and $s|D|$ is the superabundance, a priori defined only by this formula. The theorem proved is the following: Let $X_1, \cdots, X_d$ be complete sections of V_d by hypersurfaces of orders $m_1, \cdots, m_d$ of S_r, general enough for

$$V_{d-i} = X_1 \cap \cdots \cap X_i \quad (i = 1, \cdots, d-1)$$

to be irreducible and non-singular; and let d_i be the deficiency of the system traced on V_{d-i} by the complete system $|D+X_1+\cdots+X_i|$; then if $m_1, \cdots, m_d$ are not less than certain lower bounds (which do not seem to be clearly specified)

$$\delta|D| = \sum_{i=1}^{d-1} (-1)^{i+1} d_i.$$

P. Du Val (London)

5634:

Gallarati, Dionisio. Sulla varietà che rappresenta le coppie di punti di due spazi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 276–279.

Preliminary statement with only outline proof of a partial converse of the following property of the V_{p+q} in S_{pq+p+q}, Segre product $S_p \times S_q$ of an S_p and an S_q: its tangent S_{p+q}'s all meet $p+2$ S_q's lying respectively in $p+2$ S_p's, every $p+1$ of which are linearly independent. Five theorems are stated which seem to be summed up in the following statement: Sufficient conditions for a differentiable V_{p+q} in S_{pq+p+q} to be $S_p \times S_q$ are that the tangent V_{p+q}'s have the above property, and that (except for $q=1$ or $p=q=2$) V_{p+q} is of differential class $C^{(2)}$. It is conjectured that the latter condition, though used in the proof except in the cases mentioned, is in fact unnecessary in all cases.

P. Du Val (London)

5635:

Guggenheimer, Heinrich. Complementi e commenti ad un teorema di Beniamino Segre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 453–456.

The exact homology sequence is applied to deduce a few elementary formulae for birational transformations of algebraic varieties.

M. F. Atiyah (Cambridge, England)

5636:

Sampson, J. H.; and Washnitzer, G. A Künneth formula for coherent algebraic sheaves. Illinois J. Math. **3** (1959), 389–402.

The following Künneth formula is proved. Let X, Y be algebraic varieties over an algebraically closed field K, and let $\mathcal{F}$, $\mathcal{G}$ denote coherent algebraic sheaves on X, Y. Then the natural map:

$$H(X, \mathcal{F}) \otimes_K H(Y, \mathcal{G}) \rightarrow H(X \times Y, \mathcal{F}^* \otimes_{\mathcal{O}^*} \mathcal{G}^*)$$

is an isomorphism, where $\mathcal{F}^*$, $\mathcal{G}^*$ are the reciprocal images of $\mathcal{F}$, $\mathcal{G}$ on $X \times Y$, $\mathcal{O}^*$ is the sheaf of local rings on $X \times Y$ and H stands for the cohomology.

There are two steps in the proof. First the theorem is proved in the local case, i.e., when X, Y are affine. This part is quite elementary. The general case is then proved by a consideration of triple complexes. The argument here is complicated but essentially standard.

M. F. Atiyah (Cambridge, England)

5637:

★Grothendieck, A. Sur les faisceaux algébriques et les faisceaux analytiques cohérents. Séminaire Henri Cartan; 9e année: 1956/57. Quelques questions de topologie, Exposé no. 2, 16 pp. Secrétariat mathématique, Paris, 1958. 73 pp. (mimeographed)

Serre has shown that if X is a projective variety and A a coherent algebraic sheaf on X, then the vector spaces $H^i(X, A)$ are finite-dimensional; furthermore, if X is defined over the complex number field, then the corresponding holomorphic sheaf cohomology space $H^i(X^h, A^h)$ is isomorphic to the previous one, and any coherent analytic sheaf on X^h is actually of the type A^h [Ann. of Math. (2) **61** (1955), 197–228; Ann. Inst. Fourier. Grenoble **6** (1955–1956), 1–42; J. Math. Pures Appl. (9) **36** (1957), 1–16; MR **16**, 953; **18**, 511, 765]. The main results of the present paper are precisely these, where X is now any complete algebraic variety. The techniques used are those of Serre, together with the definition of sheaf cohomology via injective resolutions of sheaves, and the notion of the direct image of a sheaf on X under a map $f: X \rightarrow Y$; many important auxiliary results appear, notably on the behavior of cohomology under proper maps.

M. Rosenlicht (Berkeley, Calif.)

5638:

Nagata, Masayoshi. On the closedness of singular loci. Inst. Hautes Études Sci. Publ. Math. **1959**, 29–36.

Soit A un anneau noethérien, $S(A)$ le schéma affine de A, ensemble des anneaux locaux de A, muni de la topologie de Zariski. On sait que le lieu singulier de $S(A)$, sous-ensemble formé des anneaux locaux non réguliers, est fermé en $S(A)$, en géométrie algébrique classique et aussi dans la géométrie sur un domaine de Dedekind assujetti à la condition de finitude pour les extensions entières, définie par l'auteur [Amer. J. Math. **78** (1956), 78–116; MR **18**, 600]. On étudie ici les conditions générales sur A pour que le lieu singulier soit fermé en $S(A)$. On dira que A est de la classe G, si quelle que soit l'extension d'anneau B, de type fini sur A, le lieu singulier de $S(B)$ est fermé en $S(B)$. On voit par exemple, en utilisant le critère jacobien généralisé dû à l'auteur, que les anneaux locaux complets sont de la classe G. La classe G est fermée pour les opérations de passage au quotient, les homomorphismes et les extensions de type fini d'anneaux. On établit alors à l'aide du théorème de normalisation pour les extensions séparablement engendrées [ibid. **80** (1958), 382–420; MR **20** #862] le résultat suivant : si A est tel que pour tout idéal premier P de A et toute extension B, intègre, finie et purement inséparable de A/P, le lieu singulier de $S(B)$ est contenu en un sous-ensemble propre fermé de $S(B)$, alors A est de la classe G et réciproquement. On construit pour tout corps donné k, de caractéristique 0, un domaine d'intégrité noethérien Δ contenant k et qui n'est pas de la classe G. Enfin on établit que si A est semi-local et si pour tout idéal premier P de A et toute extension entière finie B de A/P, supposée purement inséparable, B est analytiquement non ramifié, alors A est de la classe G.

J. Guérindon (Rennes)

5639:

Nagata, Masayoshi. A general theory of algebraic geometry over Dedekind domains. III. Absolutely irreducible models, simple spots. Amer. J. Math. **81** (1959), 401–435.

Soient M un modèle sur un anneau de Dedekind I, L son corps de fonctions, A_i des modèles affines recouvrant M, et $\mathfrak{o}_i$ leurs anneaux [cf. Nagata, même J. **78** (1956), 78–116; **80** (1958), 382–420; MR **18**, 600; **20** #862]. Étant donnés un anneau de Dedekind I^* contenant I et un idéal premier $\mathfrak{p}^*$ de $L \otimes_I I^*$ tels que $\mathfrak{p}^* \cap I = (0)$, les modèles affines correspondant aux anneaux $\mathfrak{o}_i^* = I^*[\mathfrak{o}_i]$ ont pour réunion un modèle M^* (sur I^*) du corps des fractions L^* de $(L \otimes_I I^*)/\mathfrak{p}^*$; ce modèle domine M et ne dépend que de M et $\mathfrak{p}^*$; si $\mathfrak{p}^* = (0)$, on l'appelle l'extension de M sur I^* et on le note $M \otimes I^*$. Si L est une extension régulière de I, on a $\mathfrak{p}^* = (0)$ pour tout I^*; on dit alors que M est absolument irréductible. On définit, avec des restrictions analogues, le produit $M \otimes M'$ de deux modèles sur I; si les corps de fonctions de M et M' sont plongés dans un même corps, le joint $J(M, M')$ est un modèle induit de $M \otimes M'$; si M et M' sont absolument irréductibles, $M \otimes M'$ est défini et est absolument irréductible.

Soient M et F deux modèles sur I, K et L leurs corps de fonctions. On suppose que $M \otimes F$ est défini, et qu'il existe n K-automorphismes s_i du corps des fonctions L^* de $M \otimes F$ et n modèles A_i de K tels que: (1) $M = \bigcup_i A_i$; (2) $s_i^{-1}s_j((A_i \cap A_j) \otimes F) = (A_i \cap A_j) \otimes F$ pour tous i, j. Alors $M^* = \bigcup_i s_i(A_i \otimes F)$ est un modèle de L^* sur I; on dit que M^* est un modèle fibré de base M et de fibre F.

Étude des localités absolument normales, et de leur comportement par produit et extension. La limite projective M' des extensions (à des anneaux convenables) d'un modèle absolument irréductible M s'appelle la "variété" de M; quand I est un corps, celle-ci se compose des points génériques des sous-variétés de M sur le domaine universel. L'ordre d'inséparabilité est défini comme une longueur, et ses propriétés classiques s'en déduisent facilement. Un cycle généralisé [resp. entier] d'un modèle absolument irréductible M est une combinaison linéaire formelle à coefficients rationnels [resp. entiers] de localités de M; un cycle est un cycle entier dont toutes les composantes sont absolument simples; un diviseur est un cycle dont toutes les composantes sont de dimension $\dim(M) - 1$; comportement par extension et projection; définition et étude du diviseur généralisé d'une fonction, et équivalence linéaire des diviseurs.

Un idéal d'un modèle M est la donnée, pour chaque $P \in M$, d'un idéal $\mathfrak{a}(P)$ de P tel que, pour tout P, il existe un modèle affine $A \subset M$ vérifiant: "en notant $\mathfrak{o}$ l'anneau de A, il existe un idéal $\mathfrak{a}$ de $\mathfrak{o}$ tel que $\mathfrak{a}(Q) = \mathfrak{a}Q$ pour tout $Q \in A$" (ceci est essentiellement un faisceau cohérent d'idéaux). Notion d'idéal principal et de diviseur localement principal. Étude de la correspondance entre les classes de diviseurs localement principaux sur M (pour l'équivalence linéaire) et les modèles fibrés sur M par des droites.

L'ensemble des localités simples [resp. absolument simples, absolument normales] d'un modèle [resp. d'un modèle absolument irréductible] est un modèle. Critère jacobien de simplicité. Étude des produits de modèles absolument non-singuliers.

P. Samuel (Urbana, Ill.)

5640:

★**Lang, Serge. Familles algébriques de jacobiennes.** Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 155, 9 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

This is a Bourbaki report on papers by Igusa [Amer. J. Math. **78** (1956), 171–199, 745–760; MR **18**, 935, 936; and "Monodromy groups connected with algebraic surfaces" (to appear)]. Among other things, a suggestion of a definition of a generalized Picard and Albanese varieties due to Tate is given.

T. Matsusaka (Evanston, Ill.)

5641:

Monk, D. The geometry of flag manifolds. Proc. London Math. Soc. (3) **9** (1959), 253–286.

Unter Flagge versteht der Verf. eine Gesamtheit von ineinander enthaltenen Teilräumen aller Dimensionen: $F_0 \subset F_1 \subset \cdots \subset F_{n-1}$ im S_n. Ein Punktmodell $F(n+1)$ für die Gesamtheit aller Flaggen des S_n erhält man leicht als linearen Schnitt des Segreschen Produkts aller Graßmannschen $G_{n;i}$ $(i = 0, \cdots, n-1)$. $F(n+1)$ hat die Dimension $\binom{n+1}{2}$ und die Ordnung $\binom{n+1}{2}!$. Verf. betrachtet nur diese vollständigen Flaggen, während der Referent [s. Burau, Rend. Circ. Mat. Palermo (2) **3** (1954), 244–269; MR **16**, 852] und Ehresmann [Ann. of Math. (2) **35** (1934), 396–443] auch die Fälle der unvollständigen Flaggen berücksichtigen, wo nicht alle Teildimensionen auftreten. In der vorliegenden Arbeit wird jedoch durch Heranziehen moderner topologischer Begriffe weit über die genannten

und andere Vorgänger hinausgegangen. Es findet sich vor allem eine vollständige Behandlung des Basisproblems und darauf beruhend ein Beweis einer Schnittpunktsformel für algebraische Teilmengen auf $F(n+1)$. Die Basismengen gab bereits Ehresmann an; sie werden durch Zusammenfassen gewisser Schubertbedingungen beschrieben, die die einzelnen Teilräume der Flagge zu erfüllen haben. Verf. beschreibt die Basismengen sehr einfach durch Permutationen $(a_0, \cdots, a_n)$ von $(0, \cdots, n)$, deren Transpositionszahl gleich der Dimension der entsprechenden Basismenge ist. Die n nur durch je eine Transposition aus $(n, \cdots, 0)$ entstandenen Permutationen $A_0, A_1, \cdots, A_{n-1}$ symbolisieren dann die Basismengen der Dimension $\binom{n+1}{2}$, und $A_0+\cdots+A_{n-1}$ ist äquivalent einem allgemeinen hyperebenen Schnitt von $F(n+1)$. Mit Hilfe dieser Symbole wird dann die genannte Schnittformel geschrieben. Setzt man $g_i=A_{i-1}-A_i$ $(1<i\leq n-1)$, $g_0=-A_0$, $g_n=A_{n-1}$, so bilden die g_i wieder eine Basis, von der die Übereinstimmung mit einer von Hirzebruch [*Neue topologische Methoden der algebraischen Geometrie*, Springer, Berlin, 1956; MR **18**, 509] angegebenen Basis gezeigt wird.

W. Burau (Hamburg)

LINEAR ALGEBRA

5642:

Taussky, Olga. A weak property L for pairs of matrices. Math. Z. **71** (1959), 463–465.

In an earlier paper [Trans. Amer. Math. Soc. **73** (1952), 108–114; MR **14**, 236] T. S. Motzkin and the author have proved that for a pair of complex $n\times n$-matrices A, B to have the property L the following condition is necessary: Let $\alpha_1, \cdots, \alpha_t$ be the different roots of A, let m_i be the multiplicity of α_i and let $\beta_i^{(\mu)}$ be a set of m_i roots of B that in the sense of L correspond to α_i. Let A be diagonable and $P^{-1}AP=A^*$ its diagonal form and $B^*=P^{-1}BP=(B_{ij})$ $(i,j=1,\cdots,t)$, where the B_{ii} are $m_i\times m_i$-matrices. Then $|xI-B_{ii}|=\prod_{\mu=1}^{m_i}(x-\beta_i^{(\mu)})$. Pairs of matrices A, B satisfying this condition will be said to have the "property $\mathfrak{l}$". For $n=2$ it is shown that $\mathfrak{l}$ implies L. Moreover, a sufficient condition for property $\mathfrak{l}$ is established: Let $\gamma_j(z)$ denote the roots of $A+zB$; then the differential coefficients $\gamma_j'(0)$ are the roots β_j of B in some order. The converse is true if these d.c.'s exist. Special consideration is given to the case $n=2$ and to pairs of normal matrices.

H. Schwerdtfeger (Montreal, P.Q.)

5643:

Mirsky, L. On a convex set of matrices. Arch. Math. **10** (1959), 88–92.

It is known that the convex hull of $\mathfrak{P}$, the set of permutation matrices, is the collection $\mathfrak{D}$ of doubly stochastic matrices, and if $y=Dx$ with $D\in\mathfrak{D}$ and if the components of y and x are arranged in monotone decreasing order to yield $\bar{y}$ and $\bar{x}$, then $\sum_{i=1}^k \bar{y}_i\leq\sum_{i=1}^k \bar{x}_i$, $1\leq k\leq n$, and $\sum_{i=1}^n \bar{y}_i=\sum_{i=1}^n \bar{x}_i$. Let $\mathfrak{P}$ be replaced by $\mathfrak{C}_0$, the set of matrices all of whose entries are 0 or 1 with at most one 1 in each row and column. Let $\mathfrak{D}$ be replaced by $\mathfrak{C}$, the collection of matrices with non-negative entries and both row sums and column sums at most one. In the last part of the first sentence retain the inequalities but relax the terminal equality to a weak inequality. Now the corresponding propositions are true. A theorem of this type has been established by Polya [Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 49–51; MR **11**, 526].

S. Sherman (Philadelphia, Pa.)

5644:

Gerstenhaber, Murray. Note on a theorem of Wielandt. Math. Z. **71** (1959), 141–142.

Wielandt [Math Z. **53** (1950), 219–255; MR **12**, 581] determined the connection between hermitian matrices and sets of $n\times n$ matrices of complex elements which form linear spaces over the real numbers such that every matrix in the space has only real characteristic roots. From his result it follows, in particular, that a linear matrix space M of dimension n^2 over the reals with only real characteristic roots is a Jordan algebra. The present author starts at the other end. He observes that M is a Jordan algebra and then derives Wielandt's result, combining known facts about Jordan algebras. Matrices of the type considered here have also been studied by P. Lax [Comm. Pure Appl. Math. **11** (1958), 175–194; MR **20** #4572].

O. Taussky-Todd (Pasadena, Calif.)

5645:

Butler, M. C. R. Module theory and the matrix equation $f(S)=T$. J. London Math. Soc. **34** (1959), 325–336.

Let T be a linear transformation of a vector space V over a field F, and let $f(t)$ be a polynomial of degree ≥ 1 in the polynomial ring $F[t]$. To find linear transformations S of V such that $f(S)=T$ is an old problem in matrix theory [see, e.g., C. C. MacDuffee, *The theory of matrices*, Springer, Berlin, 1933; N. H. McCoy, Amer. J. Math. **57** (1935), 491–502; D. E. Rutherford, Proc. Edinburgh Math. Soc. **3** (1932–33), 135–143]. The author solves this problem under either of the following conditions: (I) V has finite or countable dimension over F, and T is locally algebraic on V; (II) T is algebraic on V. For an arbitrary linear transformation X of V the vector space V can be considered as an $F[t]$-module V_X if we define $g(t)x=g(X)x$ for $x\in V$ and $g(t)\in F[t]$. Under condition (I) or (II) V_T is a torsion module over the principal ideal domain $F[t]$, and V_T is determined to within isomorphism by its Ulm invariants. The paper's results on the solvability and solutions of the equation $f(X)=T$ are obtained in terms of the Ulm invariants of V_T and V_X.

A. Kertész (Debrecen)

ASSOCIATIVE RINGS AND ALGEBRAS

5646:

Veldkamp, G. R. Rings and fields. Nieuw Tijdschr. Wisk. **46** (1958/59), 266–289. (Dutch)

Continued from same Tijdschr. **44** (1956/57), 35–66, 207–235 [MR **20** #4545].

5647:

Vandiver, H. S. On the use of the equivalence symbol and parentheses symbols in associative distributive algebra. Math. Mag. **33** (1959/60), 13–20.

In this paper the author makes additional comments on

the notion of equivalence as he had introduced it as part of a general theory in a series of papers entitled "A development of associative algebra and an algebraic theory of numbers" [Math. Mag. **25** (1952), 233–250; **27** (1953), 1–18; MR **14**, 348; **15**, 202; with M. W. Weaver, ibid., **29** (1956), 135–151; **30** (1956), 1–8; MR **17**, 825; **18**, 465]. He remarks that "equality" is an undefined concept, subject to certain postulates involving algebraic manipulations. The property of being "unequal" is then capable of independent definition. It results from this that these two properties are not necessarily mutually exclusive and this, in turn, has the consequence that indirect proofs may be impossible in such a system. Some logical consequences of these matters are briefly referred to.

H. W. Brinkmann (Swarthmore, Pa.)

5648:

Cohn, P. M. On the free product of associative rings. Math. Z. **71** (1959), 380–398.

A ring R is the free product of a family of rings R_α, where α ranges over some index set, if (i) R is generated by the R_α, (ii) every relation in R follows from relations in the R_α and (iii) any two R_α intersect in a common subring Λ. The author gives conditions for the existence of a free product. Let Λ be a ring with unit element. Let U be a right Λ-module and U' a submodule. Let V be a left Λ-module. If for a given V the sequence $0 \to U' \otimes_\Lambda V \to U \otimes_\Lambda V$ is exact for all U, U' then V is said to be left-flat. If U and U' are given and this sequence is exact for all V, then U' is said to be right-pure in U. A ring containing Λ and having the same unit element as Λ is called a Λ-ring.

The author's main results are: (I) Given a family of Λ-rings R_α, if each R_α/Λ is left-flat then the free product R of the R_α exists and R/Λ is left-flat. The corresponding statement with "left" replaced by "right" also holds. (II) If R is a Λ-ring and if for every Λ-ring S the free product of R and S exists then Λ is both right- and left-pure in R. Using results of M. Auslander [Proc. Amer. Math. Soc. **8** (1957), 658–664; MR **19**, 390], it is shown that the free product of Λ-rings exists if Λ is regular, in particular if Λ is a division ring. Several theorems on free and universal products are given; also the modifications necessary to take care of the case of rings without unit element.

W. E. Jenner (Lewisburg, Pa.)

5649:

Warner, Seth. Characters of Cartesian products of algebras. Canad. J. Math. **11** (1959), 70–79.

Let R be a commutative ring with identity. A character of an R-algebra E is a homomorphism from E onto R. If (E_α) is a family of R-algebras indexed by a set A and if E is the Cartesian product of the E_α, then for every β in A and every character v_β of E_β, $v_\beta \circ pr_\beta$ is a character of E where pr_β is the projection homomorphism from E onto E_β. Question (1). Is every character of E of this form? Question (2). If each E_α is R, i.e., if $E = R^A$, is every character of E of this form? The author first uses an extension theorem of Buck [J. Indian Math. Soc. (N.S.) **14** (1950), 156–158; MR **12**, 796] to show that (1) has an affirmative answer if (2) does. He shows that (1) has an affirmative answer if R is an infinite field and the set A admits no Ulam measure, i.e., a non-zero, countably additive set function λ, defined on all subsets of A, taking only the values 0 and 1, such that $\lambda(F) = 0$ whenever F is a finite subset of A. This result is obtained as a corollary to a density theorem concerning a suitable weak uniform structure on the set of characters of R^A. Under the assumption that each E_α has an identity this result was obtained independently by Bialynicki-Birula and Żelasko [Bull. Acad. Polon. Sci. Cl. III **5** (1957), 589–593; MR **19**, 526]. The author shows that if R is finite and A is infinite, question (2) has a negative answer, but if R is a principal domain having at least two non-associated extremal elements (e.g., R is the integers) and A either admits no Ulam measure or has cardinality not greater than that of R, then question (2) has an affirmative answer. The author makes two applications of these results. First, the only compact principal domains are finite fields and valuation rings of locally compact fields whose topology is given by a discrete valuation of rank 1. Second, let T be a topological space and $\mathfrak{C}(T)$ the algebra of real-valued continuous functions on T. The statement that there exist no Ulam measures at all is shown to be equivalent to the statement that if T is any space such that $\mathfrak{C}(T)$ is the Cartesian product of indecomposable real algebras then every component of T is open.

K. M. Hoffman (Cambridge, Mass.)

5650:

Goldman, A. J. A note on algebras. Amer. Math. Monthly **66** (1959), 795–796.

Let A be an n-dimensional associative algebra over a field F. The left [right] regular representation of A into the algebra of $n \times n$ matrices over F is faithful if and only if A has no nonzero left [right] annihilators. It is also shown that the weakest sufficient condition so far known [C. C. MacDuffee, *An introduction to abstract algebra*, Wiley, New York, 1940; MR **2**, 241] according to which A has at least one element which is no right [left] zero-divisor is not necessary. Proofs are short and neat.

A. Kertész (Debrecen)

5651:

Beaumont, R. A.; and Wisner, R. J. Rings with additive group which is a torsion-free group of rank two. Acta Sci. Math. Szeged **20** (1959), 105–116.

Let G be a torsion-free abelian group of rank 2, D (= the nucleus of G) the set of all rationals r such that $g \in G$ implies $rg \in G$, and $\mathfrak{R}$ an associative ring on G. $\mathfrak{R}$ is non-commutative if and only if the multiplication is defined by the rule $xy = \xi(x)y$ or $xy = \xi(y)x$ for $x, y \in G$ where ξ is a non-trivial homomorphism of G into D. $\mathfrak{R}$ contains no proper divisors of zero if and only if $\mathfrak{R}$ is isomorphic to a subring of a quadratic extension of the rational number field.

L. Fuchs (Budapest)

5652:

Wong, E. T.; and Johnson, R. E. Self-injective rings. Canad. Math. Bull. **2** (1959), 167–173.

The paper is connected with the investigations of G. D. Findlay and J. Lambek [same Bull. **1** (1958), 77–85, 155–167; MR **20** #888]; it recasts some results of R. E. Johnson [Proc. Amer. Math. Soc. **2** (1951), 891–895; MR **13**, 618] on the quotient ring of a ring. The R-module M is defined as a rational extension of the R-module C if (i) $M \supset C$, and (ii) if $M \supset B \supset C$ and $f \in \mathrm{Hom}_R(B, M)$ with $f(C) = 0$ then $f = 0$. A right ideal A of the ring R is called large if $xR \cap A \neq 0$ for every nonzero x in R. $F_R(C)$ denotes the

set of all elements of the R-module C whose orders are large right ideals of R. A ring Q is called a right ring of quotients of a ring C if $Q \supset C$ and Q is a rational extension of C, both Q and C being considered as right C-modules. The main result of the paper is the following theorem: If C is a ring, C is a rational extension of R as a right R-module and $F_R(C)=0$, then C can be imbedded into a right self-injective ring S where S is a regular ring (in the sense of von Neumann [Proc. Nat. Acad. Sci. U.S.A. **22** (1936), 707–713]) and the right maximal ring of quotients of C. It is also shown that a ring has a unique left-right maximal ring of quotients. (Each ring considered is supposed to have an identity element and each R-module is supposed to be a unitary right module.)

A. Kertész (Debrecen)

5653:

van Leeuwen, L. C. A. On the zeroid radical of a ring. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 428–433.

An element b of an associative ring R is said to be l-related to the ideal $\mathfrak{a}$ if there exists an element $r \in R$ such that $r \notin \mathfrak{a}$ and $br \in \mathfrak{a}$; an ideal $\mathfrak{b}$ of R is l-related to $\mathfrak{a}$ if every element of $\mathfrak{b}$ is l-related to $\mathfrak{a}$. It is shown that in the set of all ideals l-related to $\mathfrak{a}$ there exist maximal ones which are necessarily primes, and then the intersection $\mathfrak{r}^{(l)}(\mathfrak{a})$ of all these primes is characterized as the union of all ideals $\mathfrak{b}$ such that $\mathfrak{b}+\mathfrak{c}$ is l-related to $\mathfrak{a}$ whenever $\mathfrak{c}$ is l-related to $\mathfrak{a}$. This $\mathfrak{r}^{(l)}(\mathfrak{a})$ is called the left $\mathfrak{a}$-radical of R, and the $\mathfrak{a}$-radical $\mathfrak{r}(\mathfrak{a})$ is defined as the intersection of $\mathfrak{r}^{(l)}(\mathfrak{a})$ with the right $\mathfrak{a}$-radical $\mathfrak{r}^{(r)}(\mathfrak{a})$. [$\mathfrak{r}(0)$ is the zeroid radical introduced by the reviewer, Acta Sci. Math. Szeged **16** (1955), 43–53; MR **17**, 8.] Finally, the left-zeroid radical $\mathfrak{r}^{(l)}(0)$ is discussed.

L. Fuchs (Budapest)

5654:

Divinsky, N. On simple, semi-radical and radical algebras. J. London Math. Soc. **34** (1959), 225–250.

A semi-radical ring is an associative ring in which $a=ab$ or $a=ba$ implies that $a=0$. Every ring which coincides with its Jacobson radical is a semi-radical ring. In this paper the author considers the algebra R of non-commutative polynomials without constant term in variables $y_0=x$, y_1, y_2, $\cdots$ with coefficients in a commutative ring K subject to the relations $x^2=0$, $y_{n-1}=y_nxy_n$, $n=1, 2, \cdots$. He proves that if K is a field, then R is a simple, semi-radical ring which does not reduce to the zero element alone. He shows incidentally that R is an example of a simple algebra whose multiplication algebra $T \neq LR$, where L and R are the algebras of left and right multiplications, respectively. The main part of the paper is devoted to the question as to whether for every pair p, q of elements of R there exist elements u and v in R such that $p \circ u = q \circ v$, where $x \circ y$ denotes quasi-multiplication $x+y-xy$ for x, y in R. If for some choice of the base field K, this condition is satisfied for all p, q in R, then the existence of a simple ring which coincides with its Jacobson radical follows as a consequence, via results of Andrunakievič [Izv. Akad. Nauk. SSSR. Ser. Math. **12** (1948), 129–178; MR **9**, 564] and Blair [Proc. Amer. Math. Soc. **6** (1955), 511–515; MR **17**, 230]. The author proves that if K is the field of two elements, if q is a monomial (that is, a product of the generators x, y_1, y_2, $\cdots$ of R), and if p is arbitrary in R, then there exist elements u and v in R such that $p \circ u = q \circ v$. As an application of the method, the author proves that there exists a radical algebra containing two non-zero elements x and y such that $x=yxy$, answering a question of Baer [Math. Z. **56** (1952), 1–17; MR **14**, 239]. The proofs are based on computations with monomials, and are too involved to be described here.

C. W. Curtis (Madison, Wis.)

5655:

Schenkman, Eugene. On a theorem of Herstein. Proc. Amer. Math. Soc. **10** (1959), 236–238.

The author gives a simplified proof of the theorem of Baxter [Proc. Amer. Math. Soc. **7** (1956), 855–863; MR **18**, 557] and Herstein [Ann. of Math. (2) **60** (1954), 571–575; Duke Math. J. **22** (1955), 471–476; MR **16**, 214; **17**, 577] that, if A is a simple associative ring which is not 4-dimensional over a field of characteristic 2 and if U is a proper Lie ideal of $[A, A]$, then U is contained in Z, the center of A (so that $[A, A] \bmod [A, A] \cap Z$ is a simple Lie ring).

R. D. Schafer (Cambridge, Mass.)

5656:

Ortiz, Vicente. Sur une certaine décomposition canonique d'un idéal en intersection d'idéaux primaires dans un anneau noethérien commutatif. C. R. Acad. Sci. Paris **248** (1959), 3385–3387.

It is shown that if $\mathfrak{a}$ is any ideal in a Noetherian ring, then of all the normal decompositions of $\mathfrak{a}$ there is one which is canonical; namely $\mathfrak{a}$ has a unique normal decomposition $\mathfrak{a}=\mathfrak{q}_1^* \cap \cdots \cap \mathfrak{q}_n^*$ such that if $\mathfrak{a}=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ is any normal decomposition then either $\mathfrak{q}_i^* \subseteq \mathfrak{q}_i$ or the exponent ρ_i^* of $\mathfrak{q}_i^*$ is strictly less than the exponent ρ_i of $\mathfrak{q}_i$ for all i. For another point of view let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the primes of $\mathfrak{a}$. Let $\bar{\rho}_i$ (for $i=1, \cdots, n$) be the minimum of all ρ_i such that there exists a normal decomposition $\mathfrak{a}=\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ with ρ_i the exponent of $\mathfrak{q}_i$. Let $\mathfrak{q}_i^*$ be the intersection of all $\mathfrak{p}_i$-primary ideals which contain $\mathfrak{a}+\mathfrak{p}_i^{\bar{\rho}_i}$ (thus $\mathfrak{q}_i^*$ is the $\mathfrak{p}$-component of $\mathfrak{a}+\mathfrak{p}_i^{\bar{\rho}_i}$). Then $\mathfrak{a}=\mathfrak{q}_1^* \cap \cdots \cap \mathfrak{q}_n^*$ and this is a normal decomposition. Finally it is shown that if $\mathfrak{a}$ is a homogeneous ideal in a ring of polynomials then the canonical primary components will be homogeneous.

D. K. Harrison (Philadelphia, Pa.)

5657:

Hammer, L. Anneaux infinis à factorisations finies. Com. Acad. R. P. Romîne **9** (1959), 233–235. (Romanian. Russian and French summaries)

Let R denote an infinite associative ring with unity, with the property that every proper factor ring of R is finite. If the intersection of maximal ideals of R is zero, then R is a commutative domain of integrity and every non-zero ideal of R may be written uniquely as a product of single-primed (einartig) primary ideals which are coprime to each other.

L. Fuchs (Budapest)

5658:

Auslander, Maurice; and Buchsbaum, David A. Corrections to "Codimension and multiplicity". Ann. of Math. (2) **70** (1959), 395–397.

The authors give the following corrections and supplements to their earlier paper [Ann. of Math. (2) **68** (1958),

625–657; MR **20** #6414]. (1) Although the proof of proposition 3.5 assumed tacitly that R has finite length, the assertion is true without this additional assumption. (2) To correct the last half of theorem 5.6, the following assumption should be added: there are elements $z_1, \cdots, z_s$ of $\mathfrak{m}$ with s = Krull-dimension of R such that rank $(E/(z_1, \cdots, z_s)E)=k$ for $k=1, \cdots, s$. However, it seems to the reviewer that it is probably better that this additional assumption be stated as follows: if $\mathfrak{p}$ is a maximal ideal of R containing the annihilator of E, then rank $\mathfrak{p}=s$. (3) The inequality symbol at the end of theorem 6.5 should be reversed. (4) The authors give another proof of theorem 3.6.

M. Nagata (Kyoto)

5659:

Auslander, M.; and Buchsbaum, D. A. On ramification theory in noetherian rings. Amer. J. Math. **81** (1959), 749–765.

Let S be a Noetherian ring, and let R be a subring of S such that the kernel $\mathscr{J}$ of the mapping $\phi: S\otimes_R S\rightarrow S$ defined by $\phi(x\otimes y)=xy$ is a finitely generated ideal in $S\otimes_R S$. Let $\mathscr{N}$ be the annihilator of $\mathscr{J}$ in $S\otimes S$, and set $\mathfrak{H}_{S/R}=\phi(\mathscr{N})$, which is called the homological different of S over R.

A prime ideal $\mathfrak{P}$ in S is called unramified if, with $\mathfrak{p}=\mathfrak{P}\cap R$, (a) $\mathfrak{p}S_\mathfrak{P}=\mathfrak{P}S_\mathfrak{P}$ and (b) $S_\mathfrak{P}/\mathfrak{p}S_\mathfrak{P}$ is a finite separable algebraic extension of $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. S is said to be unramified if (a) every prime ideal in S is unramified and (b) for each prime ideal $\mathfrak{p}$ in R there are only a finite number of prime ideals $\mathfrak{P}$ in S such that $\mathfrak{p}=\mathfrak{P}\cap R$.

One of the fundamental results is stated as follows: A prime ideal $\mathfrak{P}$ in S is unramified if and only if $\mathfrak{P}$ does not contain $\mathfrak{H}_{S/R}$ (theorem 2.7).

In § 3, the relationship between $\mathfrak{H}_{S/R}$ and the usual different $\mathfrak{D}_{S/R}$ is discussed, and the results are the following. (1) If R is a Noetherian normal ring and if S is a finite integral extension of R, then $\mathfrak{H}_{S/R}$ is contained in $\mathfrak{D}_{S/R}$, and (2) if furthermore S is a projective module, then $\mathfrak{H}_{S/R}=\mathfrak{D}_{S/R}$. As an application, the authors prove the purity of branch loci in regular local rings of Krull-dimension 2, which was proved independently by Serre (unpublished) and generalized by the reviewer [review below]. In § 4, some rather easy sufficient conditions for S to be a projective module are given. In § 5, the authors give some homological considerations. Though they say in the introduction that perhaps the most striking result obtained here is that if S is unramified over R and is R-projective and if T is a regular ring of finite Krull dimension containing R then $S\otimes_R T$ is regular provided that it is Noetherian, the reviewer will point out here that this result is never striking and is rather easy even if we do not assume that T is of finite Krull dimension. For, $S\otimes T$ is unramified over T, whence the regularity of T implies the regularity of $S\otimes T$ if $S\otimes T$ is Noetherian.

M. Nagata (Kyoto)

5660:

Nagata, Masayoshi. On the purity of branch loci in regular local rings. Illinois J. Math. **3** (1959), 328–333.

Let R be a noetherian, commutative ring (with unit) contained in a ring S which is a finitely generated R-module. A prime ideal $\mathfrak{p}$ in R is said to be unramified in S if $S\mathfrak{p}/\mathfrak{p}S\mathfrak{p}$ is a direct sum of separable field extensions of $R\mathfrak{p}/\mathfrak{p}R\mathfrak{p}$. S is said to be unramified over R if every prime ideal $\mathfrak{p}$ in R is unramified in S. The following abstract generalization of the purity of the branch locus in algebraic geometry [see O. Zariski, Amer. J. Math. **62** (1940), 187–221; MR **1**, 102] is established. Let R be a regular local ring and S the integral closure of R in a finite-dimensional separable field extension of the field of quotients of R. If every prime ideal of rank 1 in R is unramified, then S is unramified over R.

The proof proceeds by successive reductions. First it is shown that it suffices to prove the theorem in the case where R is complete. Then using a result of Chow [see Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 580–584; MR **20** #3150] it is shown that it suffices to prove the result in the case R has dimension 2. This was done independently by Auslander and Buchsbaum [see preceding review] and Serre (unpublished).

M. Auslander (Waltham, Mass.)

5661:

Nagata, Masayoshi. Note on coefficient fields of complete local rings. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 91–92.

A theorem of I. S. Cohen [Trans. Amer. Math. Soc. **59** (1946), 54–106; MR **7**, 509] is used to prove that if R and R' are complete local rings (which need not be noetherian), if R contains a field of characteristic $p\neq 0$, and if $R'^p\leq R\leq R'$, then there is a coefficient field of R that can be extended to a coefficient field of R'. An example is given to show that even when certain necessary conditions are satisfied, the result does not hold if the condition $R'^p\leq R$ is replaced by the condition $R'^{p^n}\leq R$, $n>1$.

H. T. Muhly (Iowa, City, Iowa)

5662:

Azumaya, Goro. A duality theory for injective modules. (Theory of quasi-Frobenius modules). Amer. J. Math. **81** (1959), 249–278.

A quasi-Frobenius module is a left A-, right B-module Q (with $a(qb)=(aq)b$) which is a faithful A-module and such that $\mathrm{Hom}_A(M, Q)$ is zero or a simple B-module for every simple A-module M. If A and B satisfy left and right minimum conditions respectively, Q is quasi-Frobenius if and only if Q is a finitely generated, injective A-module containing all simple A-modules (and then $B=\mathrm{Hom}_A(Q, Q)$). A ring A with left minimum condition is quasi-Frobenius if and only if A is a quasi-Frobenius (A, A)-module.

For every left A-module M define a right B-module $M^*=\mathrm{Hom}_A(M, Q)$. If M satisfies both chain conditions (and Q is quasi-Frobenius) then M^* is the only B-module which can be orthogonally paired with M to Q. There is also the standard duality between submodules of M and M^*. This kind of duality is, in a sense, the only possible one, but, as indicated in an addendum, this result and several related ones were anticipated by Morita [Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **6** (1958), 83–142; MR **20** #3183].

If Q is quasi-Frobenius, its A-socle and its B-socle coincide and, in fact, so do the A- and B-homogeneous components of the socle. If each such homogeneous component has equal A- and B-dimensions, Q is called Frobenius. If A and B satisfy minimum conditions, Q is Frobenius if and only if Q as an A-module is finitely generated and isomorphic to the injective hull of A modulo its radical. Section 4 also exploits the concept of

injective hulls in other ways, especially in connection with quasi-Frobenius rings.

In the final section, A is a Φ-algebra (Φ a commutative ring with minimum condition) and $Q=\mathrm{Hom}_\Phi(A, F)$, where F is the injective hull of Φ modulo its radical. Then $M^*\cong\mathrm{Hom}_\Phi(M, F)$ and reduces to the usual dual module if Φ is semisimple. *D. Zelinsky* (Evanston, Ill.)

5663:

Kurata, Yoshiki. Some remarks on quasi-Frobenius modules. Osaka Math. J. **10** (1958), 213–220.

The paper adheres throughout to the reviewer's paper [#5662 above], and aims to show that several theorems for quasi-Frobenius rings obtained by Nakayama [Ann. of Math. (2) **42** (1941), 1–21; MR **2**, 344] can be extended to quasi-Frobenius modules. For example, the author proves that if A and A^* are rings with identity element which satisfy the left and the right minimum conditions respectively, and if Q is a faithful two-sided A-A^*-module, then in order that Q be quasi-Frobenius either of the following conditions is necessary and sufficient: (1) for any irreducible left A-submodule L and irreducible right A^*-submodule R of Q, the annihilator relations $l_Q(r_{A^*}(L))=L$ and $r_Q(l_A(R))=R$ hold; (2) for any primitive idempotent elements e and e^* of A and A^*, $er_Q(N)$ and $l_Q(N^*)e^*$ are A^*-irreducible and A-irreducible respectively, where N and N^* are the radical of A and A^* respectively. *G. Azumaya* (Evanston, Ill.)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 5588, 5655, 5694.

5664:

Oehmke, Robert H. On flexible algebras. Ann. of Math. (2) **68** (1958), 221–230.

By following the methods Albert used in Trans. Amer. Math. Soc. **69** (1950), 503–527 [MR **12**, 475] to obtain results for commutative power-associative rings and algebras, the corresponding results for flexible power-associative rings are obtained. It is proved that a simple flexible strictly power-associative ring A whose characteristic is prime to 6 and with two orthogonal idempotents whose sum is not a unity element of A has the property that the associated ring A^+ is Jordan. That is, A is a non-commutative Jordan ring. If A is a finite-dimensional algebra and if A is semisimple, then A has a unity element and A is the direct sum of simple algebras. Every simple non-commutative Jordan algebra A of degree $t>1$ has A^+ a simple Jordan algebra. Combining these results with results of Albert in the above-mentioned paper and with results of R. D. Schafer [Proc. Amer. Math. Soc. **6** (1955), 472–475; MR **17**, 10], the main structure theorem states that a simple flexible strictly power-associative algebra over a field F of characteristic $\neq 2, 3$ is (a) a commutative Jordan algebra; (b) a quasi-associative algebra; (c) an algebra of degree 2; or (d) an algebra of degree 1. *L. A. Kokoris* (Chicago, Ill.)

5665:

Luchian, Tudora. A classification of real linear algebras of order 2, with divisors of zero. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 21–38. (Romanian and Russian summaries)

The contents of the paper are as stated in the title. A detailed study is made of right unity elements in these nonassociative algebras. *R. D. Schafer* (Cambridge, Mass.)

5666:

Jacobson, N. Some groups of transformations defined by Jordan algebras. I. J. Reine Angew. Math. **201** (1959), 178–195.

The Lie algebra of derivations of the exceptional Jordan algebra M_3^8 over an algebraically closed field of characteristic 0 or over the field of reals is the exceptional Lie algebra F_4 of Killing-Cartan, and the Lie algebra of linear transformations on M_3^8 leaving a certain cubic form defined on M_3^8 invariant is the exceptional Lie algebra E_6 [Chevalley and Schafer, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 137–141; MR **11**, 577; Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Math. Inst. Rijksuniv., Utrecht, 1951; MR **13**, 433]. This paper is the first of a series in which the author will extend these results and their group analogues to arbitrary central simple Jordan algebras.

The author defines, for any finite-dimensional strictly power-associative algebra A with 1, a generic norm $N(a)$ by means of the minimal polynomial of the generic element of A. Various properties of the generic minimal polynomial and the generic norm are established, including the result that, if A is a central simple Jordan algebra, then $N(\{aba\})=(N(a))^2N(b)$, where $\{abc\}=(ab)c+(bc)a-(ac)b$.

Henceforth let A be a central Jordan algebra. The group $L(A)$ is defined to be the group of (1-1) linear transformations on A leaving the generic norm invariant. It is proved that (for characteristic $\neq 2, 3$) the automorphism group $G(A)$ of A is the subgroup of $L(A)$ of elements η such that $1^\eta=1$. This permits a determination, from known results of Ancochea [Ann. of Math. (2) **48** (1947), 147–154; MR **8**, 301] and the author [Amer. J. Math. **70** (1948), 317–326; MR **9**, 564], of $L(A)$ for the various types of central simple special Jordan algebras A.

For characteristic 0 the Lie algebra analogous to the group $L(A)$ is related to the derivation algebra of A. In a final section the author's methods are applied to obtain the known structure of orthogonal groups in 4-dimensional spaces relative to quadratic forms with square discriminants. *R. D. Schafer* (Cambridge, Mass.)

HOMOLOGICAL ALGEBRA

See also 5636, 5662.

5667:

Matlis, Eben. Applications of duality. Proc. Amer. Math. Soc. **10** (1959), 659–662.

It is proved, firstly, that the finitistic left injective dimension of a left Noetherian ring is equal to its finitistic right weak dimension. The proof uses a lemma stating that inj. $\dim_R A\leq n$ holds for a left module A over a ring R if and only if $\mathrm{Ext}_R^{n+1}(R/I, A)=0$ for every left ideal I of R, a similar lemma for w. dim, and two duality isomorphisms in Cartan and Eilenberg, *Homological algebra*

[Princeton Univ. Press, 1956; MR 17, 1040; Chap. 6, Prop. 5.1 and 5.3]. The first of the last isomorphisms and the second lemma are combined then with Hattori's [J. Math. Soc. Japan 9 (1957), 381–385; MR 20 #854b] result, giving a new proof to Kaplansky's [J. Indian Math. Soc., forthcoming] theorem that an integral domain R is a Prüfer ring if and only if the torsion submodule of every finitely generated R-module is a direct summand.

T. Nakayama (Nagoya)

5668:

Nakayama, Tadasi. Note on fundamental exact sequences in homology and cohomology for non-normal subgroups. Proc. Japan Acad. 34 (1958), 661–663.

The author points out that the fundamental exact sequences in homology and cohomology of groups which describe a certain relationship between homology or cohomology groups of a group, its normal subgroup, and the factor group, may be extended to the case of non-normal subgroups. The definitions of some of the groups occurring in these sequences, as well as the proofs of the assertions, are to appear in a subsequent publication.

E. Matlis (Evanston, Ill.)

GROUPS AND GENERALIZATIONS

See also 5651, 5690, 5831, 5928.

5669:

Nagornyĭ, N. M. Beispiel einer Gruppe mit nicht rekursivem Zentrum. Z. Math. Logik Grundlagen Math. 4 (1958), 304–308. (Russian. German summary)

A set μ of words in an alphabet A is said to be recursive, if there is a normal algorithm [cf. A. A. Markov, *Teoriya algorifmov*, Trudy Mat. Inst. Steklov. no. 42 (1954); MR 17, 1038] $\mathfrak{F}$ in A which when applied to any word P, produces the empty word if and only if $P \in \mu$. Using Novikov's example of a group with insoluble word problem [P. S. Novikov, Izv. Akad. Nauk SSSR. Ser. Mat. 18 (1954), 485–524; Trudy Mat. Inst. Steklov. no. 44 (1955); MR 17, 706], the author constructs a group with non-recursive centre. The proof is based on the fact that in a free product of two non-trivial groups, an element belongs to the centre if and only if it is equal to the unit-element [cf. also G. Baumslag, W. W. Boone and B. H. Neumann, Math. Scand. 7 (1959), 191–201].

P. M. Cohn (Manchester)

5670:

Mihaĭlova, K. A. The occurrence problem for free products of groups. Dokl. Akad. Nauk SSSR 127 (1959), 746–748. (Russian)

For the definition of the occurrence problem and some relevant results see the author's preceding note [same Dokl. 119 (1958), 1103–1105; MR 20 #6454]. In the present note she gives an outline of the proof of the following theorem: If the (strong or weak) occurrence problem is soluble in two groups, then also in their free product.

K. A. Hirsch (London)

5671:

Struik, Ruth Rebekka. On verbal products of groups. J. London Math. Soc. 34 (1959), 397–400.

Let F be the free product of the groups $G_1, \cdots, G_m$ and $f(F)$ an arbitrary fixed complex commutator built from the terms of the lower central chain of F. The paper investigates $f(F) \cap [G_i]^F$, where $[G_i]^F$ is the normal closure of $[G_i]=\{[G_i, G_j], i\neq j\}$ in F. Generalizing some previous results of the author [Trans. Amer. Math. Soc. 81 (1956), 425–452; MR 17, 1051], the results lead to a general class of verbal products [S. Moran, Proc. London Math. Soc. (3) 6 (1956), 581–596; J. London Math. Soc. 33 (1958), 237–245; MR 20 #3908, #3910], which turns out to contain many distinct verbal products.

A. Kertész (Debrecen)

5672:

★Fuchs, L. Abelian groups. Publishing House of the Hungarian Academy of Sciences, Budapest, 1958. 367 pp.

The title of this book marks its content as well as the point of view from which the author looks at the subject. "Abelian groups" are considered as an independent part of mathematics, not as a special case of group-theory, nor are groups regarded as a special case of "groups with operators". The huge amount of results presented in this work—much of it due to the Hungarian school of mathematicians and in particular to the author—seems to justify this limitation, which provides the book with a proper style and even with its own grammar. "Group" is always used in the sense of Abelian group, the group composition is denoted by + and a consistent system of notations is applied throughout the book. A table of notations at the beginning and a subject index at the end give sufficient information about the notations even to those readers who are not reading the book from cover to cover. Whereas the author avoids all rhetoric, he does not spare words for giving information about the interconnection of the various problems. The proofs are given in full; sometimes several essentially different proofs are given for important theorems. It is a very readable book. Each chapter concludes with a large list of exercises—altogether more than five hundred—and some unsolved problems. In the preface, the author has expressed the hope that these problems will help to promote the research on Abelian groups. Indeed some of them have been solved in the meantime. At the end of this review, a list of solutions will be given.

The chapters I–V present what the author calls the rudiments of the theory. In Ch. I, the notions of height, direct sum, complete direct sum, cyclic and quasicyclic groups and operator modules are introduced. At this place the author states: "In extending the results to this case, there arise no real difficulties, so that such a generalisation cannot be considered as an essential one." The notion of operator module is applied only in §§ 44, 45, 53 and some exercises marked by a dot. Discussion of the additive groups of the rational numbers and of the p-adic integers. Definition of rank and of reduced rank.—Ch. II deals with the direct sums of cyclic groups and their subgroups. Two proofs of the theorem of Frobenius and Stickelberger. Criteria for the existence of a basis. Order of an element relative to an independent set. Principal systems.—Ch. III. Divisible groups D: Extension of a homomorphism of a subgroup of G into D to a homomorphism of G into D. Solution of systems of linear equations over D.—Ch. IV. Direct summands and pure subgroups. The notions of pure subgroup, bounded pure subgroup, generalized pure subgroup and neat subgroup stand in the

centre of the discussions of this chapter. Moreover algebraically compact groups are introduced without reference to topology: G is algebraically compact if it is a direct summand of every H which contains G as a pure subgroup.—Ch. V. Basic subgroups of a p-group and quasi-basis (a notion due to the author). Upper and lower basic subgroups. Theorems of Baer, Kulikov, Kovács, Szele. Every basic subgroup of a p-group G is an endomorphic image of G.

The chapters VI to VIII form—in the author's wording—the focal point of the whole development: The torsion groups (p-groups), the torsion-free groups and the mixed groups.—Ch. VI. If H is the group consisting of the elements of infinite height of a p-group G, then G/H is isomorphic to a pure subgroup of G which lies between a given basic subgroup B and a maximal torsion subgroup $\bar{B}$ of a particular complete direct sum determined by B. A complete enumeration of the pure subgroups between B and $\bar{B}$ would solve the problem of the p-groups without elements of infinite height but is difficult. The case $G=\bar{B}$ furnishes the closed p-groups. Theorems of Kulikov about closed p-groups. The theorems of Prüfer, Ulm and Zippin yield a complete characterization of all countable reduced p-groups by a matrix of countably many rows and columns. The Ulm-sequences. Example of non-isomorphic non-countable groups with the same Ulm-sequence. Applications. Direct decomposition of p-groups.—Ch. VII. Torsion-free groups. The lattice of the types of torsion-free groups. Torsion-free groups of rank 1 (R. Baer). Indecomposable groups and completely decomposable groups. Torsion-free groups over the p-adic numbers. (Here an operator domain is used as group). Slender groups (unpublished results of J. Łoś). Homogeneous and separable groups (R. Baer).—Ch. VIII is a report on the struggle of mathematicians for knowledge about the mixed groups.

The chapters IX to XVI are independent each from the others. Ch. IX. Homomorphism groups and endomorphism rings. Homomorphism, endomorphism, exact sequences. Homomorphisms of direct sums of cyclic groups and of torsion groups. The ring of endomorphisms of a group, in particular of a p-group. Rings of endomorphisms which are skewfields, Artinian rings, regular rings, torsion-rings and torsion-free rings. Automorphism groups. Characteristic subgroups.—Ch. X. Group extension: The method of factor sets (Schreier) and the method of homomorphism (Eilenberg, MacLane) are both explained and discussed. The influence of the properties of the groups L, K on $\mathrm{Ext}(L, K)$ (results of R. Baer).—Ch. XI. Tensor products. "The reader will recognize that tensor products have properties which are, in a certain sense, dual to those of homomorphism groups."—Ch. XII. The additive group of rings. Here "ring" is used in a wider sense: the multiplication may be non-associative. The first general problem is the determination of all rings which belong to a given additive group (theorems concerning the cases of torsion groups and torsion-free groups). The second kind of problems search for the additive groups which belong to a particular class of rings; e.g., the nil-groups belong only to the zero-ring and the quasi-nil groups to a finite number each of non-isomorphic rings. Quasi-nil groups are discussed in the case of torsion groups, where a simple result is obtained, as well in the other cases. Torsion-groups which admit but a countable set of non-isomorphic (associative) rings are necessarily quasi-nil. Addition groups of Artinian, semisimple and regular rings. Theorem of Szele about the additive group of a nilpotent ring with maximum condition.—Ch. XIII. The multiplication group of fields: Theorems concerning fields which are finite over the prime field (Skolem) and algebraically closed fields.—Ch. XIV. The lattice of subgroups. The lattice $L(G)$ of the subgroups of an Abelian group G is complete and modular; it is distributive when G is locally cyclic; it is complemented when G is elementary. The isomorphism between $L(G)$ and $L(H)$ implies a relation, called projectivity, between G and H. Results of R. Baer concerning projectivity in the cases of torsion-groups, torsion-free groups and mixed groups.—Ch. XV. Decomposition into direct sums of subsets. The components of this decomposition are arbitrary subsets of a group G. E.g., $G=S+T$ when T is a subgroup and S a full non-redundant system of representatives of the cosets of T. Hajos has proved that when a finite group is the direct sum of cyclic subsets, at least one of them is a group. Variants to Hajos' theorem. Decomposition into an infinite number of components.—Ch. XVI. Various questions. A. Hereditarily generating systems. These are infinite generating systems of a group G, the elements not being necessarily different, with the property that every subset of the same power is also a generating system of G. Necessary and sufficient conditions for the existence of hereditarily generating systems.—B. Universal homomorphic images. If $\mathfrak{G}(G)$ is the system of all the homomorphic images of G, then $U \in \mathfrak{G}(G)$ is a universal homomorphic image of G if to every $H \in \mathfrak{G}(G)$, U contains a subgroup isomorphic to H. Results due to the author.—C. Universal subgroups. A subgroup Z of G is called universal when every subgroup of G is a homomorphic image of some subgroups of Z. Conditions for the existence of a universal subgroup.—D. A combinatorial problem solved by M. Hall, Jr.

The Bibliography is very large; it contains also references to literature not used in the text and it extends even slightly into the realm outside the limit which the author has put up for his work. He is aware that this limitation is open to criticism. Some mathematicians still like to approach group theory in connection with the trend of ideas which started from Dedekind and they will miss an important link. A work on group theory is not due to contain a history of modern mathematics, but those young scholars who study with enthusiasm this highly informative and well written volume should not believe that the master has told them the whole thing.

Of the 86 unsolved problems proposed in the book, the following are known to have been solved in the meantime.

Prob. 11 by H. Leptin.

Prob. 20, 21 and (partly) 46 by L. Fuchs [Acta Math. Acad. Sci. Hungar. 10 (1959), 133–140; MR 21 #3482].

Prob. 24, countable case, by S. Balcerzyk [Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 141–142; MR 21 #7245].

Prob. 25 by M. Král and E. Sąsiada.

Prob. 26 by E. Sąsiada [#5673 below].

Prob. 27, 28, and 35 (=Kaplansky's Test Problem III) by E. Sąsiada.

Prob. 39 by E. Sąsiada [Ann. Univ. Sci. Budapest Sect. Math. 2 (1959), 65–66].

Prob. 42 by E. Sąsiada and D. K. Harrison.

Prob. 49 for p-groups, $p>3$, by H. Leptin.

Prob. 56 by L. Fuchs [Ann. Univ. Sci. Budapest Sect. Math. 2 (1959), 5–23] and D. K. Harrison.

Prob. 77 by A. D. Sands.

Prob. 79 follows from results of A. D. Sands [Acta Math. Acad. Sci. Hungar. 8 (1957), 65–86; MR **19**, 529].

F. W. Levi (Berlin)

5673:

Sąsiada, E. Proof that every countable and reduced torsion-free Abelian group is slender. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 143–144. (Russian summary, unbound insert)

A torsion-free abelian group H is slender if every homomorphism of a complete direct sum of countably many infinite cyclic groups into H sends almost all components into the zero of H. The result is stated in the title.

L. Fuchs (Budapest)

5674:

Sąsiada, E. On the isomorphism of decompositions of torsion-free Abelian groups into complete direct sums of groups of rank one. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 145–149. (Russian summary, unbound insert)

If G is a complete direct sum of subgroups of the rationals, then G may be written as the complete direct sum of groups $G^{(\tau)}$ where $G^{(\tau)}$ is the complete direct sum of groups of rank 1 and of fixed type τ, while τ ranges over the set Ω of all possible types. It is shown that the groups $G^{(\tau)}$ $(\tau\in\Omega)$ are up to isomorphisms uniquely determined by G. (This solves a problem of A. G. Kurosh.) If $G^{(\tau)}$ is reduced and its cardinality is less than the first cardinal of non-zero measure, then the number of components in any decomposition of $G^{(\tau)}$ as the complete direct sum of groups of rank 1 is the same. For divisible $G^{(\tau)}$ the analogous question is equivalent to the problem whether or not $2^{\mathfrak{m}}=2^{\mathfrak{n}}$ implies the equality of the cardinals $\mathfrak{m}$ and $\mathfrak{n}$.

L. Fuchs (Budapest)

5675:

Venkatarayudu, T.; and Raghavacharyulu, I. V. V. Reduction of solvable groups. Proc. Indian Acad. Sci. Sect. A **50** (1959), 196–201.

Let G be a finite group, B a normal Abelian subgroup of G, and $b^{(r)}$ $(r=1, 2, \cdots)$ the irreducible representations of B over the complex field. For each r, let $G^{(r)}$ consist of those elements of G which transform $b^{(r)}$ into itself, and let $S^{(r)}$ be the set of irreducible representations of $G^{(r)}$ which occur as components of the representation induced in $G^{(r)}$ by $b^{(r)}$. Then the sets $S^{(r)}$ are disjoint and the representations induced in G by the members of their union form a complete system of inequivalent irreducible representations of G.

R. Steinberg (Los Angeles, Calif.)

5676:

Rusakov, S. A. Subgroups of strongly Π-solvable and strongly Π-separable groups. Dokl. Akad. Nauk SSSR **127** (1959), 270–271. (Russian)

Let Π be a nonempty set of primes, let $\mathfrak{G}$ be a finite group of order mn, where m is the maximal Π-Sylow divisor of the order of $\mathfrak{G}$. If $\mathfrak{G}$ contains subgroups of all orders of the form m_in, where m_i is an arbitrary divisor of m, then $\mathfrak{G}$ is called a $\Pi\delta$-group; $\mathfrak{G}$ is called a $\Pi\Delta$-group if every subgroup of $\mathfrak{G}$ is a $\Pi\delta$-group. Relationships between the class of $\Pi\Delta$-groups and the class of strongly Π-solvable groups are cited. Sufficient conditions are given for such groups to possess a series of characteristic subgroups for which the indices have the form of a product of a number relatively prime to m and a primary factor of m. The group $\mathfrak{G}$ is called strongly separable relative to Π if every index in some normal series of $\mathfrak{G}$ is divisible either by no prime belonging to Π or by only the first power of but one prime in Π. Results generalize previous work by MacLain [Proc. Cambridge Philos. Soc. **53** (1957), 278–285; MR **19**, 13], by Zappa [Rend. Sem. Mat. Roma **4** (1938), 323–330], and by Baer [Illinois J. Math. **1** (1957), 115–187; MR **19**, 386].

R. A. Good (College Park, Md.)

5677:

Komaki, Yûji. On a generalization of certain Chunikhin's theorem on Π-factorization of finite groups. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **6**, 252–255 (1959).

Let Π be a set of prime natural numbers; a natural number $n>1$ is called Π-prime if no prime divisor of n belongs to Π. If h is a Π-block product of a finite group G, then every solvable Π-subgroup of G with order dividing h is contained in a subgroup of G with order hc, where c is some Π-prime number. The title of the paper refers to work in Mat. Sb. (N.S.) **39 (81)** (1956), 465–490; **43 (85)** (1957), 49–66 [MR **19**, 13; **20** #1708].

R. A. Good (College Park, Md.)

5678:

Ree, Rimhak. On generalized conjugate classes in a finite group. Illinois J. Math. **3** (1959), 440–444.

Sei G eine endliche Gruppe mit dem Homomorphismus σ. Die Elemente a und b heissen σ konjugiert, wenn $a=x^{-1}bx^{\sigma}$. Hierdurch sind die Elemente von G in σ-Klassen unterteilt. σ-invariant heisst die Teilmenge S von G, wenn mit x auch x^{σ} in S liegt. Der Hauptsatz besagt, dass die Anzahl der σ-Klassen gleich der Anzahl der σ-invarianten Klassen von konjugierten Elementen aus G ist. Am einfachsten wird dies nach einem Korrekturzusatz mittels einer Überlegung von Hirosi Nagao bewiesen. Der ursprüngliche Beweis des Verf. bringt die Anzahl der σ-Klassen in Zusammenhang mit den Charakteren, indem er einen Satz von I. D. Ado [Mat. Sb. (N.S.) **36 (78)** (1955), 25–30; MR **16**, 672] verallgemeinert.

J. J. Burckhardt (Zürich)

5679:

★**Higman, G. Le problème de Burnside.** Colloque d'algèbre supérieure, tenu à Bruxelles du 19 au 22 décembre 1956, pp. 123–128. Centre Belge de Recherches Mathématiques. Établissements Ceuterick, Louvain; Librairie Gauthier-Villars, Paris; 1957. 293 pp. 250 francs belges.

The paper gives a brief and clear account of the state of the Burnside problem in 1956. There is a bibliography of 9 items. The main features of the paper are likely to remain useful for some time to come. These are: (i) a neat proof of the theorem of I. N. Sanov [Leningrad. Gos. Univ. Uč. Zap. Mat. Ser. **10** (1940), 166–170; MR **2**, 212] that groups of exponent 4 are locally finite {especially helpful to those who cannot read Russian} and (ii) an indication of the viewpoint and main results of an important paper of P. Hall and the author [Proc. London Math. Soc. (3) **6** (1956), 1–42; MR **17**, 344].

{Reviewer's remarks: (1) On p. 125, line 6, S_2 should be

S_q. (2) The following more recent papers would essentially bring the present bibliography up to date: A. I. Kostrikin, Izv. Akad. Nauk SSSR Ser. Mat. **23** (1959), 3–34; M. Hall, Illinois J. Math. **2** (1958), 764–786 [MR **21** #1345]; and #5680 below.} *R. H. Bruck* (Ospedaletti)

5680:

Novikov, P. S. On periodic groups. Dokl. Akad. Nauk SSSR **127** (1959), 749–752. (Russian)

The present note is a brief description of methods leading to the proof of the following result: If $n \geq 72$, the free (Burnside) group A_n of exponent n on two or more generators is infinite. This completely settles the (unrestricted) Burnside problem for every $n \geq 72$.

The final paper is expected to be very long indeed. The present note, though clearly written, necessarily leaves a few things to the imagination, and the reviewer cannot guarantee that his interpretation is always the correct one.

Let n, q be natural numbers, fixed throughout the discussion and satisfying the inequalities $12 \leq q \leq n/6$. Consider the set of all (associative) words on a specified finite set of (two or more) generators. Call a word W a Dyck word if $W=1$ is a relation in A_n. Call a word of form X^t (where X is a word and t is a natural number) periodic with period X, exponent t. (In particular, all periodic words of exponent n are Dyck words.) For a word A of form BC, where B, C are nonempty words, call B, C segments of A. Call a word uncancellable if it cannot be shortened by cancellation. With these definitions we may state the author's final theorem as follows.

Theorem 11: Each nonempty, uncancellable Dyck word contains a segment which is a periodic word with an exponent not less than $q/2$.

This theorem is used in the following way: Since $q \geq 12$, a word containing no periodic segment with exponent 6 or more cannot be a Dyck word. {The paper has a misprint here: 72 instead of 6.} However, by a paper of S. E. Aršon [Mat. Prosvešč., no. 2, **24** (1934)], one can construct an infinite sequence on three symbols which contains no periodic segment with an exponent as large as 2. Consequently, in the case of three or more generators, A_n is infinite for $n \geq 72$. The author states that, using Aršon's methods, he has constructed an infinite sequence on two symbols which contains no periodic segment with an exponent as large as 3. Therefore, in the two-generator case as well, A_n is infinite for $n \geq 72$.

The role of the number q appears in the definition of Ω_q-transformations. The latter are of two types, circular and linear. A circular Ω_q-transformation is a word transformation which either replaces a word AX^tB (where $q \leq t \leq n-q$) by the word $AX^{t-n}B$ or circularly permutes words. A linear Ω_q-transformation is a word transformation which replaces a word AX^tB (where $q \leq t \leq n-q$ as before if A, B are both empty, and $q/2 \leq t \leq n-q$ in the contrary case) by the word $AX^{t-n}B$. The concept of a periodic word is generalized to that of a conditionally periodic word. A product-word $X_1 \cdots X_t$ is conditionally periodic if there exist words $X, u_1, v_1, \cdots, u_t, v_t$ such that, for each i, X_i can be transformed by a circular Ω_q-transformation into u_iXv_i, and such that $v_iu_{i+1}=1$ for $1 \leq i < t$ and $v_tu_1=1$. In this case, the words X_i are called the conditional periods and the natural number t is called the exponent of the conditionally periodic word.

A word A is called reduced provided (i) A is uncancellable, (ii) no segment of A can be transformed into 1 by a circular Ω_q-transformation, (iii) no segment of A is equal to a conditionally periodic word with exponent exceeding $n-q$. For each word X, let $R(X)$ denote the class of all reduced words into which X can be transformed by circular Ω_q-transformations. Theorem 1 states that, for each X, $R(X)$ is nonempty (though its only member may be the empty word 1). Theorem 2 states that, for each X, if $R(X)$ contains a nonempty word, each member of $R(X)$ is nonempty.

A conditionally periodic word $X_1 \cdots X_t$ is called essential if each conditional period X_i is a nonempty reduced word and if t is maximal subject to this condition.

Two essential conditionally periodic words with the same exponent, $X_1 \cdots X_t$ and $Y_1 \cdots Y_t$, are called equivalent if there exist words $u_1, v_1, \cdots, u_t, v_t$ such that $u_iY_iv_i$ lies in $R(X_i)$ and $u_iv_{i+1}=1=u_tv_1$. {The paper has $R(v_i)$ instead of $R(X_i)$.} This definition leads to the equivalence class, $L(X_1 \cdots X_t)$, of an essential conditionally periodic word $X_1 \cdots X_t$.

Theorems 3, 4 state that inversion and cyclic permutation, respectively, preserve the class of essential conditionally periodic words, and that inversion preserves exponents. Theorem 5 states that a conditionally periodic word of exponent t contains a periodic word with an exponent not less than the smaller of t and $q/2$.

Theorems 6 and 7 (which we omit) show in particular that the class of all essential conditionally periodic words of exponent at least q can be partially ordered as follows: We say that the essential conditionally periodic word B precedes the essential conditionally periodic word A provided (i) the classes $L(A)$ and $L(B)$ are distinct, (ii) A has form A_1CA_2, (iii) B has form B_1CB_2, (iv) C contains at least q conditional periods of B.

At this stage the author is able to give a system of defining relations for A_n as follows. Theorem 8: The set of all equations $X_1 \cdots X_n=1$, where $X_1, \cdots, X_n$ are arbitrary essential conditionally periodic words, forms a set of defining relations for the group A_n.

According to the author, the following theorem may be deduced from the work of Tartakovskiĭ [Izv. Akad. Nauk SSSR. Ser. Mat. **13** (1949), 483–494; Mat. Sb. (N.S.) **30** (**72**) (1952), 39–52; MR **11**, 493; **13**, 819].

Theorem 9: Let a group be defined by a finite or infinite number of relations of form $A_{1i}A_{2i} \cdots A_{ti}=1$, $t \geq 6$. Assume that the class of words $A_{1i} \cdots A_{ti}$ is closed under cyclic permutation and inversion. Assume also that neither of two words A_{ri}, A_{sj} can completely disappear under cancellation in the product word $A_{ri}A_{sj}$. Then every nonempty uncancellable Dyck word A can be obtained by multiplying together the words $A_{1i} \cdots A_{ti}$ in such a manner that, after cancellation, at least one of the segments A_{ki} remains in the word A.

For the group A_n the author states a sharper result and gives a brief indication of the proof: (Theorem 10) Every nonempty uncancellable Dyck word of the group A_n can be obtained by multiplying together essential conditionally periodic words $A_{1i} \cdots A_{ni}$ with exponent n and conditional periods A_{ji} in such a manner that every two conditionally periodic words which occur in the product cancel against each other at most q conditional periods.

He then states that theorems 5, 9, 10 imply theorem 11.

{Reviewer's remarks. (1) The reviewer would like to suggest, to those who may feel that the theory centering around the Burnside problem is now at an end, that the

present paper constitutes merely an important early chapter in a chronicle which might be entitled "What makes a group finite?" (2) The reviewer was much assisted by a German translation of this note prepared by H. Salzmann.} *R. H. Bruck* (Frankfurt a.M.)

5681:

Albert, A. A.; and Thompson, John. Two-element generation of the projective unimodular group. Illinois J. Math. **3** (1959), 421–439.

$\mathfrak{F}$ is the field of p^m elements, $\mathfrak{M}$ is the multiplicative group of all n-by-n matrices with elements in $\mathfrak{F}$ and determinant 1, $\mathfrak{N}$ is the normal subgroup of $\mathfrak{M}$ of scalar matrices $\rho\mathbf{I}$ with $\rho^n=1$. The quotient group $\mathfrak{S}$, $=\mathfrak{M}/\mathfrak{N}$, is the projective unimodular group over $\mathfrak{F}$. The main theorem is:

$\mathfrak{S}$ is generated by two elements $A\mathfrak{N}$, $B\mathfrak{N}$, i.e., $\mathfrak{M}$ is generated by A, B and $\lambda\mathbf{I}$. The coset $A\mathfrak{N}$ has period 2.

In the general case

$$A=\begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & \mathbf{I}_{n-4} & & \\ & \alpha & & 1 & \\ 1 & & & & 1 \end{bmatrix},\quad B=(-1)^n\begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & -1 & & \\ & & & \mathbf{I}_{n-4} & \\ 0 & 0 & \delta & & 1 \\ 1 & k & 0 & & 0 \end{bmatrix}$$

where α is a primitive element of $\mathfrak{F}$, $2\delta=\alpha$, $2k+1=p$. Separate forms and proofs are given for the various special cases of low values of p, m, n that require them.

T. G. Room (Sydney)

5682:

Room, T. G. The generation by two operators of the symplectic group over GF(2). J. Austral. Math. Soc. **1** (1959/61), part 1, 38–46.

If $\Gamma=(a_{ij})$ is a $2m+2$-square matrix with $a_{ij}=1$ $(i\neq j)$, $a_{ii}=0$, a Clifford set is defined by the columns of each matrix A such that $A^t\Gamma A=\Gamma$ over GF(2). Such matrices form a group which contains Γ. Dickson's $A(2m, 2)$ is isomorphic to the group of automorphisms of that group which transforms Clifford sets into Clifford sets. The author gives generators Q, R for $A(2m, 2)$ which contains a group generated by Q and Γ isomorphic to S_{2m+2} [cf. Room and Smith for the generators of $A(2m, p)$, Quart. J. Math. Oxford Ser. (2) **9** (1958), 177–182; MR **20** #3917].

G. de B. Robinson (Toronto, Ont.)

5683:

Newman, M.; and Reiner, I. Inclusion theorems for congruence subgroups. Trans. Amer. Math. Soc. **91** (1959), 369–379.

Let $G_t=\mathrm{GL}_t^+[J]=\mathrm{SL}_t[J]$ be the group of $t\times t$ matrices with rational integral elements and determinant 1. The set of matrices $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ with $\dim A=r$, $0<r<t$, $C\equiv 0 \bmod n$, $n>1$, forms a proper subgroup $G_r(n)$. Moreover, if $n|m$, then $G_r(n)\supset G_r(m)$. Theorem 1: If $G_r(n)\supset H\supset G_r(m)$, then $H=G_r(q)$ for some q, $n|q$, $q|m$. The set of matrices above with $\dim A=r$, $C\equiv 0 \bmod n$, $B\equiv 0 \bmod m$, forms a subgroup $G_r(m, n)$. Theorem 2: Suppose $t=2r$, $(m, n)=1$, $G_r(m, n)\subset H\subset G_{2r}$. Then $H=G_r\,(m_1, n_1)$ with $m_1|m$, $n_1|n$. Both theorems 1 and 2 are generalized in various directions, and extended to the symplectic group.

J. L. Brenner (Palo Alto, Calif.)

5684:

Conrad, Paul. Right-ordered groups. Michigan Math. J. **6** (1959), 267–275.

A right-ordered group (ro-group) is a group with a full order relation that is invariant under right translations. After some simple criteria for a group to admit a right ordering, and for a right-ordering to be an ordering (i.e., invariant also under left translations) have been derived, the extensions of a partial right-ordering to a full right-ordering and the order homomorphisms of ro-groups are studied. It is well known that if in an ordered group C and C' are convex subgroups and C' covers C then C is normal in C' and C'/C is order isomorphic to an additive group of real numbers. The corresponding proposition is shown not to be true in ro-groups in general, but to be equivalent to the following condition: to every pair a, b of positive elements there is a positive integer n such that $(ab)^n>ba$. Further results extend to ro-groups some completion lemmas proved by the author for ordered groups [Proc. Amer. Math. Soc. **5** (1954), 323–328; MR **15**, 849]. Finally the author shows that every extension of an ro-group by an ro-group can be right ordered, that the cartesian (or unrestricted direct) product of ro-groups can be right-ordered, and that the group of all order automorphisms of a fully ordered set can be right-ordered. [For this last result, proved by the same argument, see also P. M. Cohn, Mathematika **4** (1957), 41–50; MR **19**, 940].

B. H. Neumann (Manchester)

5685:

Ribenboim, Paulo. Sur quelques constructions de groupes réticulés et l'équivalence logique entre l'affinement de filtres et d'ordres. Summa Brazil. Math. **4** (1958), 65–89.

$\mathfrak{F}$ étant l'ensemble des filets d'un groupe réticulé (abélien) G et $\mathfrak{F}'$ un treillis contenant convenablement $\mathfrak{F}$, l'auteur montre qu'il existe un isomorphisme de G dans un groupe réticulé totalement décomposable G' induisant l'immersion $\mathfrak{F}\subset\mathfrak{F}'$. Il en déduit que pour toute algèbre de Boole $\mathfrak{B}$ il existe un groupe réticulé dont le treillis des filets est isomorphe à $\mathfrak{B}$. L'auteur se sert de ce résultat pour montrer l'équivalence logique des deux affirmations suivantes (conséquences toutes deux de l'axiome du choix): $(\mathscr{U}')$ Si J est un filtre d'une algèbre de Boole, il existe un ultrafiltre plus fin que J. $(\mathscr{O}')$ G étant un groupe abélien réticulé, il existe sur G un préordre plus fin (noté $\leqq$), total et saturé, compatible avec la structure de groupe et tel que: $a\leqq b$ et $a\leqq c$ entraînent $a\leqq\inf(b, c)$. On sait que $(\mathscr{U}')$ entraîne $(\mathscr{U})$ et que $(\mathscr{O}')$ entraîne $(\mathscr{O})$ ainsi définis: $(\mathscr{U})$ Tout filtre sur un ensemble est contenu dans un ultrafiltre. $(\mathscr{O})$ Tout ordre sur un ensemble admet un ordre total plus fin. On sait que $(\mathscr{U})$ entraîne $(\mathscr{O})$, mais on ne sait pas si $(\mathscr{O})$ entraîne $(\mathscr{U})$.

P. Jaffard (Paris)

5686:

Hall, P. Periodic FC-groups. J. London Math. Soc. **34** (1959), 289–304.

Periodic FC-groups are groups in which every element has finite order and finitely many conjugates. They have been much studied under various names, particularly in relation to Sylow theorems, by R. Baer [Duke Math. J. **6** (1940), 598–614; MR **2**, 2], P. A. Gol'berg [Mat. Sb. (N.S.) **19** (1946), 451–460; MR **8**, 367] and others, especially of the school of Kuroš, who calls them "locally normal".

Here they are investigated in relation to (restricted) direct products of finite groups. The main results are that the countable periodic FC-groups are precisely the factor groups of subgroups of countable direct products of finite groups, and that the subgroups of countable direct products of finite groups are precisely those countable periodic FC-groups which are residually finite (that is, representable as subgroups of cartesian products of finite groups). Countability is essential here, for it is also shown by an ingenious example that a group can be periodic (of exponent 4) and FC (in fact with derived group of order 2, which ensures that the group is nilpotent of class 2 and that the classes of conjugate elements are boundedly finite, the bound on their cardinals being 2) and yet fail to be a factor group of a subgroup of a direct product of finite groups; the order of the group is the cardinal of the continuum. Incidental results include the following: There is a subgroup of a countable direct product of finite groups which is not itself a direct product of finite groups, and not even a factor group of such a direct product; the example constructed is nilpotent of class 2, whereas by a result of L. Koulikoff [ibid. **16** (1945), 129–162; MR **8**, 252] no similar abelian example can exist. Every subgroup of a countable direct product of finite groups can be written as the product of two (in general overlapping) normal subgroups each of which is a direct product of finite normal subgroups of the whole group. Every countable FC-group is a factor group of a subgroup of a countable direct product of finite groups and infinite cyclic groups; and if its centre is trivial, then it is a subgroup of a countable direct product of finite groups. Finally it is shown that every periodic FC-group is the union of a sequence of marginal subgroups M_n ($n=1, 2, \cdots$) defined in terms of the laws valid in the symmetric group S_n of degree n.

B. H. Neumann (Manchester)

5687:

Černikov, S. N. Some classes of groups with conditions for finiteness. Dokl. Akad. Nauk SSSR **127** (1959), 1176–1178. (Russian)

The author announces, without proofs, a number of theorems that generalize known results previously obtained by himself and by other representatives of the Russian group-theoretical school. We cite only the first three. The remaining ones require a number of further definitions and are of a rather special character. An account of these will be given when the full proofs are available.

An H-group shall be an infinite group in which every infinite factor-group of an infinite normal subgroup has proper normal subgroups. This class comprises that of the locally soluble groups. If an infinite group contains normal H-subgroups, then the product of all these is also an H-group, the unique maximal normal H-subgroup. Theorem 1: If an infinite group has an ascending normal series whose infinite factors are all H-groups, then the group itself is an H-group. Theorem 2: No H-group can contain only a finite number of conjugacy classes. [For the case of SI-groups, in particular locally soluble groups, see Černikov, Mat. Sb. (N.S.) **13 (55)** (1943), 317–333; MR **6**, 201.] A C-group shall be an infinite group in which every countable subgroup has an ascending normal series whose factors satisfy max (maximal condition for subgroups). This class is strictly wider than that of the groups with an ascending normal series whose factors satisfy max—example by Kargapolov. Infinite locally nilpotent groups are C-groups. Again, every infinite group containing normal C-subgroups has a unique maximal normal C-subgroup. Theorem 3: Every locally finite C-group satisfying $\min_{ab}$ has an abelian normal subgroup of finite index satisfying min and, therefore, satisfies min. [For a similar result on locally soluble groups see Černikov, ibid. **27 (69)** (1950), 185–200; **28 (70)** (1951), 119–129; MR **12**, 477; they cover, in particular, the case of C-groups with ascending normal series with locally nilpotent factors. For the case of C-groups with ascending normal series in which only the finite factors need not be locally nilpotent see Plotkin, Trudy Moskov. Mat. Obšč. **6** (1957), 299–336; MR **19**, 529; where such groups are called WF-groups.]

K. A. Hirsch (London)

5688:

Artzy, R. Crossed-inverse and related loops. Trans. Amer. Math. Soc. **91** (1959), 480–492.

Un C.I.L. est un quasigroupe Q, avec unité bilatère, u, satisfaisant pour tout x, $y \in Q$, $xx'=u \Rightarrow (xy)x'=y$. Ces C.I.L. ont déjà été étudiés, au point de vue des isomorphismes, dans deux précédents papiers de l'A. [Riveon Lematematika **8** (1954), 81; Proc. Amer. Math. Soc. **6** (1955), 448–453; MR **16**, 670, 1083]. Si $xx'=u$, l'application $(x\to x')$, $(xy)'=x'y'$ s'appelle "automorphisme inverse" et l'élément $\langle g, f\rangle$ de l'ensemble produit Q^2, l'élément de translation de l'isotopie ($\xi=(x\to xg)$, $\eta=(y\to fy)$, $\zeta=1$). Théorème: Dans un C.I.L., si un élément appartient à un des nuclei de Rees, il appartient au centre et aux deux autres nuclei. Lemme: Si un isotope principal, G^*, d'un C.I.L., avec les éléments de translation g et f, est encore un C.I.L., alors f et g satisfont (α), pour tout x, y, $(fg)(xy)=(fx)(gy)=[f(xy)]g=f[g(xy)]$ et réciproquement, et de plus $G^*\simeq G$. De ces identités, l'A. déduit une série d'autres; les calculs sont conduits au moyen des translations à droite, R, et à gauche, L, de G. {On peut faire remarquer que ces identités sont deux à deux duales comme on le voit en les établissant par des équations directes entre les éléments de G.} L'isotopie $\{\xi=(x\to fx), \eta=(y\to gy), \zeta=(z\to fgz)\}$ est une autotopie si g et f satisfont à la condition que $ag * fb=ab$ soit C.I.L.; la paire ordonnée $\langle g, f\rangle$ s'appelle le "companion-couple" de l'autotopie. L'ensemble des valeurs de f satisfaisant (α) est le "nucleus-companion", C. Il coïncide avec les nuclei-companions procurés par g et par gf. L'ensemble des éléments de C qui sont à la fois centraux et associatifs est un sous-quasigroupe normal dans C. La fin du paragraphe est consacrée aux quasigroupes avec élément neutre satisfaisant $(xy)(zx)=[x(yz)]x$, qui ont fourni l'occasion d'une multitude d'exemples et d'une variété de théorèmes élémentaires par démarquage des groupes. Mais l'A. a su donner un intérêt à ces structures par l'emploi de la notion de compagnons. Le 2e paragraphe concerne les conditions relatives à l'ordre de certains quasigroupes finis [cf. T. Ikuta, Nat. Sci. Rep. Lib. Arts Fac. Shizuoka Univ. no. 9 (1956), 1–2; MR **19**, 1037]. Le 3ème donne pour les C.I.L. infinis une construction qui est un élargissement de la méthode déjà exposée par l'A. dans Proc. Amer. Math. Soc. **6** (1955), 448–453 [MR **16**, 1083]. Page 485, dans la preuve du Théorème 4, dernière expression de θ, lire U^{-1}, au lieu de U.

A. Sade (Marseille)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 5710, 5901, 5950, 6003.

5689:

Loonstra, F. Topologisch subdirekte Produkte. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 434–438.

A topological group G is called a subdirect product of a family of topological groups $\{G_\alpha\}$, if: (i) for each α, there is an open continuous homomorphism φ_α from G onto G_α; (ii) for each element $g \neq e$ of G, there is an α such that $\varphi_\alpha(g) \neq e_\alpha$, where e and e_α denote the identity elements of G and G_α respectively. For a topological group F, G is said to be a F-subdirect product of $\{G_\alpha\}$, if: (a) for each α, there is an open continuous homomorphism ψ_α from G_α onto F; (b) G is isomorphic to the subgroup of the direct product $\prod_\alpha G_\alpha$ formed by all elements $\{g_\alpha\}$ such that $\psi_\alpha(g_\alpha)$ is independent of α. If G is a F-subdirect product of $\{G_\alpha\}$ for some topological group F, then clearly G is a subdirect product of $\{G_\alpha\}$. Given a compact group G which is a subdirect product of $\{G_\alpha\}$, the main theorem gives necessary and sufficient conditions in order that G be a F-subdirect product of $\{G_\alpha\}$ for some F. In particular, if a compact group G is a subdirect product of two topological groups G_1, G_2, then G is also an F-subdirect product of G_1, G_2 for some suitable F. In the case of discrete groups, this last result has been obtained previously by L. Fuchs [Acta Math. Acad. Sci. Hungar. **3** (1952), 103–120; MR **14**, 612]. When a compact Abelian group G is a F-subdirect product of Abelian groups $\{G_\alpha\}$, the author determines the annihilator of G in the character group X of $\prod_\alpha G_\alpha$ and thereby expresses explicitly the character group of G as a quotient group of X. *Ky Fan* (Notre Dame, Ind.)

5690:

MacBeath, A. M.; and Świerczkowski, S. On the set of generators of a subgroup. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 280–281.

After proving an algebraic theorem on the generators of a subgroup, the authors deduce two corollaries, of which the second is: If G is a locally compact group generated by a compact set and H is a closed subgroup of G such that G/H is compact, then H is generated by a compact set. This generalizes a result of Poincaré that if G is finitely generated and G/H is finite, then H is finitely generated. *A. M. Gleason* (Cambridge, Mass.)

5691:

Tsuji, Kazô. On Haar measures in subgroups. Bull. Kyushu Inst. Tech. Math. Nat. Sci. **5** (1959), 13–17.

Let $\mathfrak{G}$ be a separable locally compact group and let G and H be closed subgroups (neither required to be normal) such that $\mathfrak{G} = GH$ and $G \cap H = \{e\}$. Then each element $s \in \mathfrak{G}$ has a unique representation $s = \xi x = y\eta$, where $x, y \in H$, $\xi, \eta \in G$. If we write $y = \hat{\xi}(x)$ and $\eta = \hat{x}(\xi)$, then this determines, for each fixed $\xi \in G$, a map $\hat{\xi}$ of H into itself, and for each fixed $x \in H$, a map $\hat{x}$ of G into itself. The author computes the effects of these maps on the Haar measures in H and G, respectively. Symbolically,

$$d\alpha(\hat{x}(\xi)) = \frac{\Delta(\hat{\xi}(x))}{\delta'(\hat{\xi}(x))}\frac{\delta(\hat{x}(\xi))}{\delta(\xi)}\, d\alpha(\xi),$$

$$d\beta(\hat{\xi}(x)) = \frac{\delta(\hat{x}(\xi))}{\Delta(\hat{x}(\xi))}\, d\beta(x),$$

where α and β are the left Haar measures on G and H, and Δ, δ and δ' are the modular functions on $\mathfrak{G}$, G and H, respectively. *A. M. Gleason* (Cambridge, Mass.)

5692:

Edwards, D. A. On independent group characters. Bull. Amer. Math. Soc. **65** (1959), 352–354.

Let H be a compact abelian group with character group H^*. The author shows that a subset S of H^* is algebraically independent if and only if the functions in S are independent, with respect to Haar measure on H, in the sense of probability theory. *K. deLeeuw* (Stanford, Calif.)

5693:

Dixmier, Jacques. Sur les représentations unitaires des groupes de Lie nilpotents. IV. Canad. J. Math. **11** (1959), 321–344.

The author continues his investigation of the unitary representations of nilpotent Lie groups [Parts I–III, Amer. J. Math. **81** (1959), 160–170; Bull. Soc. Math. France **85** (1957), 325–388; Canadian J. Math. **10** (1959), 321–348; MR **21** #2705; **20** #1928, #1929], restricting his attention to the group G_n of $n \times n$ real matrices (ξ_{jk}) having $\xi_{jj} = 1$ for $1 \leq j \leq n$ and $\xi_{jk} = 0$ for $1 \leq j < k \leq n$. He determines the center of the enveloping algebra of its Lie algebra, its principal series of irreducible unitary representations (which are induced by characters of certain commutative subgroups), the corresponding infinitesimal characters [in the sense of part II] and global characters [in the sense of Godement, J. Math. Pures Appl. **30** (1951), 1–100; MR **13**, 12], and a Plancherel formula involving these representations. *K. deLeeuw* (Stanford, Calif.)

5694:

Veldkamp, F. D. Note on the real forms of a simple Lie algebra. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 300–303.

The characteristic δ of a real simple Lie algebra is the signature of its Killing form. In the list of real simple Lie algebras given by E. Cartan [Ann. Sci. École Norm. Sup. (3) **31** (1914), 263–355], one notices that for the type D_n, where n is a square, there are two real forms which give the same characteristic $-n$. In this paper it is proved that these two real forms are actually distinct. *R. Ree* (New York, N.Y.)

5695:

Seligman, George B. On automorphisms of Lie algebras of classical type. Trans. Amer. Math. Soc. **92** (1959), 430–448.

Let $\mathfrak{L}$ be a Lie algebra of classical type over an arbitrary field F of characteristic different from 2 or 3. Then $\mathfrak{L}$ has a commutative Cartan subalgebra $\mathfrak{H}$ such that $\mathfrak{L}$ and $\mathfrak{H}$

satisfy the axioms (i)–(v) in the paper by Mills and Seligman [J. Math. Mech. **6** (1957), 519–548; MR **19**, 631]. A commutative Cartan subalgebra $\mathfrak{H}$ satisfying axioms (iii)–(v) is called a standard Cartan subalgebra. For any basis element e_α for a root space $\mathfrak{L}_\alpha$ of a non-zero root of $\mathfrak{L}$ with respect to a standard Cartan subalgebra, $\sigma = \exp(\mathrm{ad}\,\lambda e_\alpha)$, $\lambda \in F$, is defined by the power series, and is an automorphism of $\mathfrak{L}$. Consider all standard Cartan subalgebras, and all possible generators e_α of root spaces L_α, $\alpha \neq 0$, associated with any one of them. The group of automorphisms of $\mathfrak{L}$ generated by all automorphisms of the form $\exp(\mathrm{ad}\,\lambda e_\alpha)$, $\lambda \in F$, forms a subgroup I of the full automorphism group called the group of invariant automorphisms. The main results of the paper are concerned with the position of I in the full automorphism group of $\mathfrak{L}$. First of all, let $\mathfrak{H}$ be a standard Cartan subalgebra, and $\alpha_1, \cdots, \alpha_r$ a fundamental system of roots of $\mathfrak{L}$ with respect to $\mathfrak{H}$. Let S_{α_i} be the Weyl reflection $\phi \to \phi - \phi(h_i)\alpha_i$, where h_i is an element of $[\mathfrak{L}_{-\alpha_i}, \mathfrak{L}_{\alpha_i}]$ such that $\alpha_i(h_i) = 2$, and $\phi \in \mathfrak{H}^*$, the dual space of $\mathfrak{H}$. The author proves, using methods similar to those of the reviewer [Trans. Amer. Math. Soc. **86** (1957), 91–108; MR **20** #933], that the group W generated by the S_{α_i} is independent of the particular fundamental system $\alpha_1, \cdots, \alpha_r$, and that if $\beta_1, \cdots, \beta_s$ is a second fundamental system, then $r = s$, and there exists a unique $S \in W$ such that $\beta_1, \cdots, \beta_r$ is a permutation of $\alpha_1 S, \cdots, \alpha_r S$. This result, in case F is the field of complex numbers, has been proved in another way by Satake [J. Math. Soc. Japan **2** (1951), 284–305; MR **14**, 448]. After reducing the problem to the case of simple Lie algebras, and automorphisms which map a standard Cartan subalgebra $\mathfrak{H}$ onto itself, the author proves the following result. Let $\alpha_1, \cdots, \alpha_r$ be a fundamental system of roots relative to $\mathfrak{H}$, whose matrix (A_{ij}) has for its determinant the natural number d. Let σ be an automorphism of $\mathfrak{L}$ such that $\mathfrak{H}\sigma = \mathfrak{H}$, and $\alpha_i^\sigma = \alpha_i$, $1 \leq i \leq r$, where σ operates on the roots in the obvious way. Suppose further that $e_{\alpha_i}\sigma = \mu_i e_{\alpha_i}$, $1 \leq i \leq r$, where each μ_i is a dth power in F. Then σ is an invariant automorphism. One of the main consequences of this result is that if F is algebraically closed, and $\mathfrak{L}$ is not of type A_r $(r \geq 2)$, D_r, $(r \geq 4)$, or E_6, then every automorphism of $\mathfrak{L}$ is an invariant automorphism. In all cases, at least when F is algebraically closed, the author asserts that the group of invariant automorphisms is a normal subgroup of finite index of the full automorphism group, and promises more precise information in subsequent papers. *C. W. Curtis* (Madison, Wis.)

5696:

★**Séminaire C. Chevalley, 1956–1958. Classification des groupes de Lie algébriques.** 2 vols. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. ii+166+ii+122 pp. (mimeographed)

This is an exhaustive study of linear algebraic groups over an algebraically closed field K of arbitrary characteristic. It contains not only a complete classification of semi-simple algebraic groups, but also many fundamental theorems on certain subgroups, homogeneous spaces, root systems, representations and their weights, and isogenies.

Before going into a detailed review, we shall describe roughly how the global classification is achieved. Unlike the classical theory of compact semi-simple groups wherein root systems are defined in terms of the Lie algebra, they are here defined directly, and turn out to be of the same types as in the classical theory. The root system R of the linear algebraic group G is a subset of the character group $X(T)$ of a maximal torus T of G, and $X(T)$ is a free abelian group of rank dim T. While the root system R of a semi-simple group G determines G locally (i.e., up to an isogeny), the pair $(R, X(T))$ not only determines G globally, but also gives complete information about what isogenies of and onto G are possible. All the possibilities of the pair $(R, X(T))$ can be determined from the relation $R \subset X(T) \subset P$, where P is the additive group generated by the weights of G with respect to T. The weights of G with respect to T are defined in an ingenious manner by using projective representations of G; they generate a subgroup of $\mathbf{Q} \otimes X(T)$, and their relation to R depends only on the type of G.

The work consists of twenty-four expositions.

Expositions 1–8 are preliminary. Algebraic varieties are defined in a manner similar to Serre [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953], replacing sheaves by local systems. Then comes the exposition of Borel's paper [ibid. **64** (1956), 20–82; MR **19**, 1195], along with the necessary algebraic geometry. The notions of affine algebraic groups (a.a.g.), semi-simple and unipotent parts of an element (or group), and the Lie-Kolchin theorem on solvable a.a.g. are introduced. A torus is an algebraic group isomorphic to K^{*n}. Maximal tori of a connected a.a.g. are conjugate. A Borel subgroup of an algebraic group G is defined to be a maximal connected solvable subgroup of G. If G is connected, then all Borel subgroups B of G are conjugate, and G/B is a projective variety. If T is a maximal torus of G and if N and C are, respectively, the normalizer and centralizer of T in G, then $W = N/C$ is finite, and is called the Weyl group of G with respect to T. Cartan subgroups and regular elements are defined purely group-theoretically, and their properties are derived. Also a general theory of homogeneous spaces is given.

Expositions 9–17 contain a global treatment of many familiar notions which are derived in the classical case from Lie algebras. A theorem, which is basic in the whole theory, asserts that a Borel subgroup of an a.a.g. is its own normalizer. The radical of an a.a.g. G is defined to be the component of the unit element of the intersection of all Borel subgroups of G, and semi-simplicity is defined as usual. In what follows, let G denote a connected a.a.g., T a maximal torus of G, B a Borel subgroup of G containing T, N the normalizer of T, C the centralizer of T, $W = N/C$, the Weyl group of G with respect to T, and define $\Gamma = \Gamma(T) = \mathrm{Hom}(K^*, T)$ to be the group of all rational homomorphisms $K^* \to T$, and similarly $X = X(T) = \mathrm{Hom}(T, K^*)$, the character group of T. Γ and X are free abelian groups of rank dim T, and there exists a nondegenerate pairing of Γ and X onto the additive group of integers. W is made to operate on Γ and X by inner automorphisms (of G) by elements in N. A torus Q or element $\gamma \in \Gamma$ is called semi-regular if Q, respectively $\gamma(K^*)$, is contained in only a finite number of Borel subgroups, and singular otherwise. The singular elements in Γ form a number of hyperplanes in Γ dividing the semi-regular elements in Γ into Weyl chambers. Each hyperplane of singular elements in Γ corresponds to a singular torus of codimension 1 in T, and conversely. Also there is one-one correspondence $B \leftrightarrow \mathfrak{C}(B)$ between Borel subgroups $\supset T$ and Weyl chambers. For each hyperplane H of singular elements there exists one and only one $w \neq 1$ in W which leaves H fixed elementwise. w

is a symmetry: $w^2=1$, and W is generated by such symmetries corresponding to the walls of any given Weyl chamber.

For nonsolvable G, roots with respect to T are defined as follows. For each singular torus Q of codim 1 in T, let Z be the centralizer of Q, and R the radical of Z. Then Z has exactly two Borel subgroups $\supset T$, and each of them can be written in the form $B \cap Z$ for a Borel subgroup $B \supset T$ of G. The radical of $B \cap Z$ is $B \cap R$, and the quotient $B^u \cap Z/B^u \cap R$ of their unipotent parts, where B^u is the unipotent part of B, is isomorphic to the additive group of K. Let $f: B^u \cap Z/B^u \cap R \to K$ be such an isomorphism. Then there exists $\alpha \in X$ such that $f(tst^{-1}) = \alpha(t)f(s)$ for $t \in T$ and $s \in B^u \cap Z/B^u \cap R$. This α is called the root associated with Q and the Borel subgroup $B \cap Z$ of Z. α is negative on the Weyl chamber $\mathfrak{C}(B)$. If we use the other Borel subgroup of Z, we obtain $-\alpha$ instead of α. We may write $Q=Q_\alpha=Q_{-\alpha}$, $Z=Z_\alpha=Z_{-\alpha}$. If we fix B (and T) and let Q vary, then we obtain all roots of G which are negative on the Weyl chamber $\mathfrak{C}(B)$. The roots corresponding to the walls of $\mathfrak{C}(B)$ are fundamental roots with respect to B. {Reviewer's note: For the possible motivation of the above definitions, see Chevalley, Tôhoku Math. J. (2) 7 (1955), 14–66 [MR 17, 457].}

In case G is semi-simple, $R=Q$ and $B^u \cap R=\{1\}$. In what follows, G will be assumed to be semi-simple. The structure of B is studied in terms of the groups $P_\alpha = B^u \cap Z_\alpha$, where α runs over roots negative on $\mathfrak{C}(B)$. As for the structure of G/B, it is shown that if $(\sigma_w)_{w\in W}$ is a set of representatives mod C of elements of W ($C=T$ for a G semi-simple) then $\{B\sigma_w B | w \in W\}$ is a partition of G. For each $B\sigma_w B$ the Bruhat decomposition is given, and is used to study the linear systems of divisors on the variety G/B. It is proved that the irreducible rational projective representations of G associated with the linear systems of divisors on G/B exhaust all irreducible rational projective representations of G. Let ρ be a rational projective representation of G in a vector space V, and $\omega: \mathrm{SL}(V) \to \mathrm{PL}(V)$ the canonical homomorphism. If $\tilde{G}$ [resp. $\tilde{T}$] is the component of the unit element of $\omega^{-1}(\rho(G))$ [resp. $\omega^{-1}(\rho(T))$], then $\tilde{T}$ is a maximal torus of $\tilde{G}$, and there is an isomorphism $\zeta: \mathbf{Q} \otimes X(\tilde{T}) \to \mathbf{Q} \otimes X(T)$ which arises naturally. Let $(v_1, \cdots, v_n)$ be a basis of V such that $\tilde{t}\cdot v_i = \chi_i(\tilde{t})v_i$ for all $\tilde{t} \in \tilde{T}$. The elements $\chi_i = \zeta(\tilde{\chi}_i)$ in $\mathbf{Q} \otimes X(T)$ are called the weights of ρ. If ρ is irreducible and if Kv_1 is invariant under $\rho(B)$, then χ_1 is called the dominant weight of ρ. Fundamental dominant weights of G are also defined, and the relations between fundamental roots, fundamental dominant weights, and the Weyl group are derived. The additive group generated by weights contains $X(T)$.

The decomposition of G arising from the decomposition of its root system into simple components is also given.

Expositions 18–24 contain the main theorems which enable us to classify all semi-simple groups. An isogeny $f: G \to G'$ is defined to be a rational epimorphism of finite kernel. With the isogeny f is associated canonically an isomorphism $\varphi: \mathbf{Q} \otimes X(T') \to \mathbf{Q} \otimes X(T)$ which has the following two properties: (1) $\varphi(X(T')) \subset X(T)$; (2) there exists a bijection ψ of the set of roots of G with respect to T onto the set of roots of G' with respect to T' such that $\varphi(\psi(\alpha)) = q(\alpha)\alpha$ for all roots α of G, where $q(\alpha)$ is a power of the characteristic of K. Now, one of the main theorems is the converse: if $\varphi: \mathbf{Q} \otimes X(T') \to \mathbf{Q} \otimes X(T)$ is an isomorphism possessing the two properties (1) and (2), above, then φ is associated with an isogeny $G \to G'$. The proof of this theorem is long and complicated. Let R and P be the additive groups generated by roots and weights, respectively, of G with respect to T. Then the group P/R is independent of G (of a given type), and $R \subset X(T) \subset P$. If $X(T)=P$, G is said to be simply connected. For any given type, a simply connected group exists and is unique up to an isomorphism; all the other semi-simple groups of the same type are images under isogenies of this simply connected group. *R. Ree* (New York, N.Y.)

5697:

Tits, Jacques. Sur la classification des groupes algébriques semi-simples. C. R. Acad. Sci. Paris **249** (1959), 1438–1440.

In this paper the author attempts a classification of semi-simple algebraic groups defined over a field K of characteristic 0. For K algebraically closed, a complete classification was given by Chevalley [review above]. Let G be a semi-simple algebraic group defined over K, and L a sufficiently large Galŏis extension, with Galois group Γ, of K. By considering the action of Γ upon certain subgroups of G defined over L, the author succeeds in defining the notion of "anisotropic part" F of G, which together with the action σ of Γ on the root system of G determines G up to a K-isomorphism. The question of the existence of G for given σ and F is reduced to that of a certain representation of F. Some results of W. Landherr [Hamburg Abh. **11** (1935), 41–64] and N. Jacobson [Duke Math. J. **4** (1938), 534–551] are recovered. Examples dealing with the types E_r ($r=6, 7, 8$) and A_r are given.

R. Ree (New York, N.Y.)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

See also 5914.

5698:

Lester, Anne. Some semigroups on the two-cell. Proc. Amer. Math. Soc. **10** (1959), 648–655.

Theorem: Let the two-cell S be given a continuous associative multiplication such that on the boundary B one has $xy=x$ for all $x, y \in B$. Suppose, moreover, that S has a zero, 0, in the interior and that there are no other interior idempotents. Then there exists an arc T from 0 to some boundary point e, such that T is a subsemigroup with identity e, and $S=BT$. Also, if $f_1, f_2 \in B$, $t_1, t_2 \in T$, then $(f_1t_1)(f_2t_2) = f_1(t_1t_2)$, and if $f_1t_1 = f_2t_2$ then $t_1=t_2$.

This theorem is related to the work of Mostert and Shields [Ann. of Math. (2) **65** (1957), 117–143; MR **18**, 809] who considered the case where the boundary B is a subgroup of S. *A. L. Shields* (New York, N.Y.)

FUNCTIONS OF REAL VARIABLES

See also 5544, 5711a, 5849.

5699:

Whitney, Hassler. On bounded functions with bounded nth differences. Proc. Amer. Math. Soc. **10** (1959), 480–481.

Complément à un résultat démontré par l'auteur [J. Math. Pures Appl. (9) **36** (1957), 67–95; MR **18**, 889]. On considère les fonctions f définies dans un intervalle fermé I, et dont les différences $\Delta_h{}^n f(x)=\sum_i (-1)^{n-i}\binom{n}{i} f(x+ih)$ sont bornées; alors il existe une constante L_n telle que si f est bornée dans un intervalle $I'\subset I$, on peut trouver un polynome P de degré $\leq n-1$, tel que $|f-P|\leq L_n \sup|\Delta_h{}^n f|$, $x\in I$.
J. Favard (Paris)

5700:

Lizorkin, P. I. Boundary properties of a certain class of functions. Dokl. Akad. Nauk SSSR **126** (1959), 703–706. (Russian)

Soit G un ouvert du plan dont la frontière Γ est une courbe assez régulière (par exemple, $\Gamma\in C^2$); soit $1<p<\infty$, $0\leq\alpha<p-1$; on désigne par ρ la distance d'un point (x,y) à Γ, et on considère les fonctions $u(x,y)$, sommables dans G, dont le gradient (au sens de Sobolev = au sens des distributions) saitisfait $\iint_G \rho^\alpha\,|\text{grad } u|^p\,dxdy<\infty$. L'auteur rappelle d'abord qu'à tout u on peut associer une fonction qui lui est presque partout égale et tend, sur presque toute normale à Γ, vers une limite $\varphi(s)$ ($s\in\Gamma$; ds = élément d'arc sur Γ). Puis il donne une condition nécessaire et suffisante pour que φ soit une telle limite, à savoir

$$\iint_{\Gamma\times\Gamma} |\varphi(t)-\varphi(s)|^p|t-s|^{\alpha-p}dsdt<\infty.$$

Les démonstrations se font d'abord quand Γ est un carré. Fautes typographiques (mélange de ρ, p et τ).
J. P. Kahane (Montpellier)

5701:

McShane, E. J. A canonical form for antiderivatives. Illinois J. Math. **3** (1959), 334–351.

Let D^p, where $p=(p_1,p_2,\cdots,p_N)$, denote the operator

$$\partial^{|p|}/(\partial x^1)^{p_1}\cdots(\partial x^N)^{p_N},\quad |p|=\Sigma p_r,$$

on real functions in R^N. A canonical form for the solutions of the equation (A) $D^pF=f$ for given f is discussed, and constructed by an extension to R^N of the theory of divided difference operators. A pseudopolynomial of degrees less than $p_1,p_2,\cdots,p_N$ is a sum of monomials $(x^j)^k g(x)$, where $k<p_j$, $g(x)$ is independent of x^j. It is proved that f is a pseudopolynomial if and only if it satisfies a certain set of difference equations. Given a function f and a set of hyperplanes

$$\text{(B)}\quad x^1=x_1{}^1,\ x^1=x_2{}^1,\cdots,\ x^1=x_{p_1}{}^1,\cdots,\ x^N=x_{p_N}{}^N,$$

no two the same, there is a unique pseudopolynomial equal to f on these hyperplanes.

If f is continuous there is a unique F satisfying (A) and vanishing on the hyperplanes (B), and if f is summable over every finite interval there is a unique F satisfying (A) and vanishing on (B) and satisfying the conditions that D^qF is continuous if $q_r\leq p_r-1$ and is an indefinite integral if $q_r=p_r-1$. Explicit formulae are given for F as an integral involving f. These results are used to prove the "Fundamental lemma of the Calculus of Variations", that if

$$\int_{a^1}^{b^1}\cdots\int_{a^N}^{b^N}[D^{p_1}\varphi_1(\xi^1)]\cdots[D^{p_N}\varphi_N(\xi^N)]\cdot f(\xi^1,\cdots,\xi^N)d\xi^1\cdots d\xi^N=0$$

whenever $\varphi_1,\cdots,\varphi_N$ are infinitely differentiable functions, φ_j vanishing near a^j, b^j, then f is a pseudopolynomial of degrees less than $p^1,p^2,\cdots,p^N$. It is also proved that all the weak solutions with continuous derivatives up to a given order of $Df=0$ are locally equal to a pseudopolynomial of degrees less than $p_1,p_2,\cdots,p_N$.
J. L. B. Cooper (Cardiff)

5702:

de Bruijn, N. G. Taylor series and similar series. Simon Stevin **33** (1959), 20–26. (Dutch)

In this lecture the author discusses two different approaches to Taylor series with remainder, and generalizes them. The method of repeated integration by parts leads to the Bürmann-Lagrange formula. [The reader who is interested in this approach should also consult Kronecker, S.-B. Preuss. Akad. Wiss. Berlin **1885**, 841–862.] The second generalization has the form $\sum_{n=0}^N q_nT_n(g)=g^{(N)}(\xi)$, where T_n are suitable linear functionals and q_n are constants. The cases $T_n(f)=f(a_n)$ and $T_n(f)=f^{(k_n)}(a_n)$ are investigated.
R. P. Boas, Jr. (Evanston, Ill.)

5703:

Garg, K. M. On a function of non-symmetrical differentiability. Gaṇita **9** (1958), 65–75.

A certain function defined by U. K. Shukla [Gaṇita **8** (1957), 81–104; MR **20** #6497] was asserted by him to be everywhere continuous but was later shown by Shukla and Garg [Gaṇita **9** (1958), 27–32; MR **21** #2714] to be in fact discontinuous at the points of a dense set. In the present paper it is shown how Shukla's function may be modified so as to become everywhere continuous while retaining certain other desired properties, namely, bounded variation and non-symmetric differentiability at the points of a dense set.
T. A. Botts (Charlottesville, Va.)

5704:

Obrechkoff, Nikola. Sur le théorème de Hermite et Poulain. C. R. Acad. Sci. Paris **249** (1959), 21–22.

This paper is essentially the same as that by the author in C. R. Acad. Bulgare Sci. **11** (1958), 5–8 [MR **21** #707]. The only addition is the application of the main result of the previous paper to obtain generalizations of the composition theorems of I. Schur and Malo.
M. Marden (Milwaukee, Wis.)

MEASURE AND INTEGRATION

See also 5544, 6015.

5705:

Volkmann, Bodo. Die Dimensionsfunktion von Punktmengen. Math. Ann. **138** (1959), 145–154.

For any set M in Euclidean n-space R_n, the Hausdorff dimension of M at the point x is defined by

$$\dim(x,M)=\lim_{\varepsilon\to 0}\dim\{M\cap K_\varepsilon(x)\},$$

where $K_\varepsilon(x)$ is the closed sphere of centre x, radius ε, and dim $\{E\}$ denotes the Hausdorff dimension of the set E in

the sense of Besicovitch. The dimension function of the set M is the real function defined for all $x \in R_n$ by $f(x) = \dim(x, M)$.

The properties of dimension functions are studied and the following proved. (i) [Theorems 1 and 2] For any set M the dimension function is strongly upper semicontinuous at each point of R_n in the sense that $f(x_0) = \lim\sup_{x \to x_0} f(x)$, but there exist strongly upper semicontinuous functions which are not dimension functions of any set. (ii) [Theorems 3 and 4] Any real function $f(x)$ defined for $x \in R_n$ which is piecewise continuous and strongly upper semi-continuous with $0 \leq f(x) \leq n$ is the dimension function of a suitable set. (iii) [Theorem 5] If T is a locally bounded transformation of E_1 onto E_2 and M_1 is a subset of E_1 with $T(M_1) = M_2$, then for every $x \in E_1$, $\dim(x, M_1) = \dim(T(x), M_2)$. Various special types of dimension function are then discussed, examples being given with a basis in number theory.

S. J. Taylor (Birmingham)

5706:

Sîmboan, G. Sur les mesures compactes. Com. Acad. R. P. Romîne **9** (1959), 105–110. (Romanian. Russian and French summaries)

Bewiesen wird ein Satz über die Approximation meßbarer Funktionen durch Treppenfunktionen, deren Konstanzmengen einem hinsichtlich des Maßes dichten Mengensystem angehören.

K. Krickeberg (Aarhus)

5707:

Sîmboan, G. Mesures dans des espaces topologiques ordonnés. Com. Acad. R. P. Romîne **9** (1959), 237–243. (Romanian. Russian and French summaries)

"The author describes the construction of a Radon measure on a locally compact ordered space, whose restriction to interiors of intervals has values given in advance." (From the author's summary.)

E. Hewitt (Seattle, Wash.)

5708:

Young, L. C. Remarks on a chapter of the integral. Rend. Circ. Mat. Palermo (2) **7** (1958), 48–54.

In § 2 of the article there is the following notation. X: metric space. Class $H(\varepsilon)$ of "ε-sets", $\varepsilon > 0$: hereditary class of subsets of X which shrinks with ε. Gauge g: nonnegative function defined on $H(\varepsilon)$ for sufficiently small ε, $g(0) = 0$, $g(A) \leq g(B)$ whenever $A \subset B$. $\gamma = \gamma(g)$: measure deduced from g by generalizing Hausdorff's procedure. g and $\gamma = \gamma(g)$ are termed "typical" if $g(E) = \inf g(O)$ for open $O \supset E$. A simple adaptation of proofs in Saks, *Theory of the integral* [Stechert, New York, 1937; pp. 53 and 72–73] leads to a formulation of Lusin's theorem for a typical measure and a σ-integrable function. In § 3 the notation is, R', R'': euclidean spaces, $R = R' \times R''$. X', X'': figures in R', R'' respectively, $X = X' \times X''$. G'': nonnegative, additive function of (closed) intervals $I'' \subset R''$ vanishing outside X''. Stieltjes G''-gauge $g'' = \mathbf{g}(G'')$: for $E'' \subset R''$, $g''(E'') = \inf G''(I'')$ for cubes I'' whose interior includes E'', $\gamma'' = \gamma(g'')$. Similar notations are occasionally used with R', G' and R, G. $g' = g'(-, t)$: typical gauge in X' depending on $t = x'' \in X''$, such that $g'(E', t)$ is a γ''-integrable function of t whenever E' is a bounded open subset of X'. For O open in X, $g(O) = \int_{X''} g'(O', t) d\gamma''(t)$, where O' = projection of O in X'. For arbitrary $E \subset X$, $g(E) = \inf g(O)$ for $O \supset E$. Theorem (a): Let $f(x)$, $x = (x', x'') \in X$, be a real-valued γ-integrable function. Then there exists a γ''-nullset N'' such that, for each $x'' \in X'' - N''$, $f(x', x'')$ is a γ'-integrable function of x', $F(x'') = \int_{X'} f(x', x'') d\gamma'(x')$ is γ''-integrable and $\int_X f(x) d\gamma(x) = \int_{X''} F(x'') d\gamma''(x'')$. Th. (b) is a corollary of Th. (a) when g' is a Stieltjes gauge $\mathbf{g}(G')$; g is then a Stieltjes gauge also, $g = \mathbf{g}(G)$. In Th. (c), g is $= \mathbf{g}(G)$, $G'(I', t)$ is defined as cube-derivative or Radon-Nikodym integrand of $G(I' \times I'')$ regarded as function of I'', and the final equation reads $\int_X f dG = \int_{X''} F dG'' + \int_N f dG$ for a suitable γ-nullset N. § 4 contains concise, sometimes scanty proofs which the reviewer could only partially check, e.g., the uniformization procedure in the γ''-measurability proof in § 4, (i). The letter W has apparently different meanings in the first and second parts of § 4, (i). The definition of G' in Th. (c) raises difficulties due to the dependence on I' of the exceptional γ''-nullset; cf. K. Krickeberg, Ann. Mat. Pura Appl. (4) **44** (1957), 105–133 [MR **20** #399], § 2.

Chr. Pauc (Nantes)

5709:

Peck, J. E. L. Doubly stochastic measures. Michigan Math. J. **6** (1959), 217–220.

If λ is a non-negative measure on a measurable space X, call μ on $X \times X$ doubly stochastic (with respect to λ) if and only if $\mu(E \times X) = \mu(X \times E) = \lambda(E)$ for all measurable E in X. Call a doubly stochastic measure π a permutation matrix on $X \times X$ if and only if π is carried on some S in $X \times X$ that represents a 1-1 correspondence between the two copies of X. The set of doubly stochastic measures is evidently convex, and almost as evidently the permutation matrices are extreme points of this convex set. In this paper it is shown that the doubly stochastic measures are the convex closure of the permutation measures under a suitable topology when λ is Lebesgue measure on a (not necessarily bounded) interval.

L. J. Savage (Chicago, Ill.)

5710:

Hulanicki, A. On subsets of full outer measure in products of measure spaces. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 331–335. (Russian summary, unbound insert)

Translation-invariant extensions of character $\mathfrak{c}$ of Lebesgue measure on the circle K were constructed by Kodaira and Kakutani [Ann. of Math. (2) **52** (1950), 574–579; MR **12**, 246] by suitably imbedding K in the torus group of dimension $\mathfrak{c}$. It is shown here that translation-invariant extensions of character $2^{\mathfrak{c}}$ can also be obtained by this method. {Extensions of character $2^{\mathfrak{c}}$ invariant under all Lebesgue measure preserving transformations have also been constructed by Kakutani and the reviewer [ibid., 580–590; MR **12**, 246] using an entirely different method.} The construction is based on the following theorem: If (X, B) is the Cartesian product of a family $\{X_t, B_t) : t \in T\}$ of measurable spaces each of which has a subset of cardinality $\leq \mathfrak{c}$ that meets every non-empty member of B_t, and if T has cardinality $\leq 2^{\aleph}$, then X contains a subset of cardinality $\leq \aleph^{\aleph_0}$ that meets every non-empty member of B. From this it follows that the torus group of dimension $2^{\mathfrak{c}}$ contains a subset with outer measure 1 whose projection on some component is 1:1 and consists of linearly independent elements. This leads to the desired invariant extension. As a further application

it is asserted that, assuming the continuum hypothesis, any compact abelian group of cardinality $\leq 2^{2^{\aleph}}$ contains a subset of outer Haar measure 1 and cardinality $\leq \aleph$.
J. C. Oxtoby (Bryn Mawr, Pa.)

5711a:

Džvaršeišvili, A. G. The Denjoy integral and some questions of analysis. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **25** (1958), 273–372. (Russian)

5711b:

Džvaršeišvili, A. G. Analytic functions in the interior of the unit circle. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **25** (1958), 373–410. (Russian)

The author takes pains to treat repeated and term-by-term integration in the Denjoy sense as a non-trivial generalization of the corresponding Lebesgue theory. Moreover, he uses a norm, taken from Stieltjes integration with respect to a continuous function, to study sequences of integrals and the space of Denjoy integrable functions. The resulting exposition is well-suited to applications to a variety of topics of analysis, and this is illustrated by his discussion of approximation in his norm by trigonometric polynomials, by his treatment of singular integrals, by his proof of the existence of a continuous function of two variables with partial derivatives given almost everywhere by a pair of arbitrary finite measurable functions, and by a number of results in the theory of Fourier series, and in the cognate theory of analytic and harmonic functions defined in the unit disc. For instance, the author characterizes the class of such functions representable by a Denjoy-Poisson or by a Denjoy-Cauchy formula and establishes for them the existence of radial limits almost everywhere on the circumference; if the radial limits vanish almost everywhere the functions are identically zero.
L. C. Young (Madison, Wis.)

5712:

Haupt, Otto; et Pauc, Christian. Mesures simplement et dénombrablement additives adaptées à une pseudo-topologie. J. Math. Pures Appl. (9) **38** (1959), 213–234.

In this paper the authors lay the foundations of a theory of measure in Boolean σ-algebras, generalizing that for locally compact spaces which they had previously developed [Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1955**, 187–218; MR **17**, 469]. In a Boolean σ-algebra r with unit R a content function j countably additive on a Boolean subring q of r is given. A pseudo-topology o (class of somas, called open, containing $\emptyset$, R, finite meets, and countable joins) is also given. j is said to be adapted to the pseudo-topology o if (1) each $Q \in q$ lies between an open soma $\underline{Q}$ and a closed soma $\bar{Q}$ such that $(\bar{Q}-\underline{Q})$ belongs to q and is contained in an open soma of arbitrarily small content, and (2) q includes an open base q_0 with the property that whenever a closed soma C contained in some member of q is covered by a subfamily t of q_0 then C is also covered by some countable subfamily of t. On the basis of these axioms a theory of extension of j to measures j' and j'' with properties corresponding to those of Baire and Borel measures is developed.
J. C. Oxtoby (Bryn Mawr, Pa.)

5713:

Matthes, Klaus. Über eine Verallgemeinerung des Lebesgueschen Integralbegriffes. III. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **7** (1957/58), 329–344. (English, Russian and French summaries)

Im ersten und zweiten Teil dieser Arbeit [dieselbe Z. **5** (1955/56), 287–295; **6** (1956/57), 221–236; MR **19**, 1042; **20** #5270] hatte der Verf. verschiedene Aspekte einer Theorie der Maße mit Werten in einem bedingt vollständigen Vektorverband V beschrieben. In der vorliegenden Mitteilung behandelt er die Integration von reellen Funktionen und, allgemeiner, Spektralscharen in einer Booleschen sigma-Algebra B hinsichtlich eines solchen Maßes. Die Stetigkeitseigenschaften des Integrals als Funktion des Integranden, d.h. die Verallgemeinerungen der Sätze von B. Levi, Lebesgue und Fatou, werden eingehend untersucht. Sodann studiert der Verf. diejenigen Maße μ, für die die Abbildung $\sigma \to \int \sigma d\mu$ ein sigma-Isomorphismus des geordneten Raums der integrierbaren Spektralscharen σ in V ist, wobei natürlich alles "modulo der Elemente aus B vom Maße Null" betrachtet werden muß. Diese Eigenschaft von μ ist gleichwertig damit, daß disjunkte Elemente aus B endlichen Maßes disjunkte Maße in V haben; in der Formulierung dieses Satzes auf S. 338 heißt es übrigens fälschlich "Homomorphismus" statt "Isomorphismus". In diesem Zusammenhang behandelt der Verf. einige Beispiele, unter anderem den Fall, daß V eine Einheit hat und B die vollständige Boolesche Algebra der Komponenten von V ist, wobei der Freudenthalsche Spektralsatz bewiesen wird. Für die vierte Mitteilung wird die Korrektur einiger Irrtümer aus den drei ersten Teilen angekündigt.
K. Krickeberg (Aarhus)

5714:

Turner, L. H. Measures induced on a σ-algebra by a surface. Duke Math. J. **26** (1959), 501–509.

Following the terminology of Cesari's book [*Surface area*, Princeton Univ. Press, 1956; MR **17**, 596], let T be a continuous BV mapping from an admissible set A into E_3, and $T_r = p_r \circ T$ where p_r is a perpendicular projection on the rth coordinate plane for $r=1, 2, 3$. For A compact, Cesari [Rend. Mat. e Appl. (5) **14** (1955), 655–673; MR **17**, 470] defined regular measures φ, φ_r, φ_r^+, φ_r^-, $r=1, 2, 3$, on the middle space which agree for open sets with area and the corresponding variations of the plane transformations T_r, respectively. The restriction that A be compact is removed in the present paper. It is also shown that φ_r^+ and φ_r^- are mutually singular.
W. H. Fleming (Providence, R.I.)

FUNCTIONS OF A COMPLEX VARIABLE

See also 5711a-b, 5843, 5846, 5867, 5868, 5873, 5874.

5715:

Trohimčuk, Yu. Yu. Conditions for monogenicity. Dokl. Akad. Nauk SSSR **127** (1959), 285–286. (Russian)

Let $w=f(z)$ be a continuous function in some domain D of the z-plane. Definitions: (1) $f(z)$ possesses the property K' in $z \in D$ if on each of three rays t_1, t_2, t_3 going

out from z and belonging to three different straight lines, there exists one and the same limit

$$\lim_{h\to 0} \operatorname{Arg} \frac{f(z+h)-f(z)}{h}, \quad z+h \in t_1, t_2, t_3.$$

(2) $f(z)$ possesses the property K'' in $z \in D$ if on each of three rays t_1, t_2, t_3 as above there exists

$$\lim_{h\to 0} \left| \frac{(f(z+h)-f(z))}{h} \right|,$$

the 3 limits being the same. (3) $f(z)$ is incompletely monogenic in D if $D=(\bigcup_n \mathscr{E}_n) \cup H$ $(n=1, 2, \cdots)$, where H is at most a denumerable set of points belonging to D and on every set $\mathscr{E}_n$ $(n=1, 2, \cdots)$ the function $f(z)$ is monogenic relatively to $\mathscr{E}_n$, i.e. for every $z \in \mathscr{E}_n$ there exists a limit

$$\lim_{h\to 0,\, z+h\in\mathscr{E}_n} \frac{f(z+h)-f(z)}{h} = f_{\mathscr{E}_n}'(z).$$

(4) Let $w=f(z)$ be a continuous mapping of D onto the w-plane in the neighborhood of $z_0 \in D$ and denote $w_0=f(z_0)$. A point $z_0 \in D$ is called a U-point of the mapping $w=f(z)$ if there exists a sequence $\{l_k\}$ $(k=1, 2, \cdots)$ of simple closed Jordan curves shrinking to z_0, each of the curves containing z_0 in its interior, but the image of the curve l_k $(k=1, 2, \cdots)$ in the w-plane through the mapping $w=f(z)$ does not contain the point w_0. (5) The mapping $w=f(z)$ is called direct in the U-point z_0 if a positive circuit of each l_k $(k=1, 2, \cdots)$ by the point z gives to the function $\operatorname{Arg}(w-w_0)$ a non-negative increase.

The author states the following theorem: Let $w=f(z)$ be a continuous function in D possessing one of the following properties: (1) in every point $z \in D$, with the exception of at most a denumerable set of them, the property K' holds; (2) in every point $z \in D$, except at most a denumerable set of them, the property K'' holds, and in almost all U-points (if such points exist) the mapping $w=f(z)$ is direct; (3) the function $f(z)$ is incompletely monogenic in D. Then $f(z)$ is analytic everywhere inside D; and if in the case (2) U-points do not exist, then $f(z)=\text{const}$.

The theorem is not true without the hypothesis that $f(z)$ is continuous. But in this case the 3 conditions of the theorem can be replaced by the Cauchy-Riemann conditions and the following theorem holds: Let $f(z)$ $(z=x+iy)$ be defined in D and satisfy in every point there the Cauchy-Riemann conditions. Then $f(z)$ is analytic in every point of D, except perhaps in a closed discrete set $p \subset D$, the projections of which on the x- and y-axes are also discrete sets. That p can be non-empty is shown by the well-known example of Montel: $f(z)=e^{-1/z^4}$ for $z\neq 0$ and $f(z)=0$ for $z=0$.

It is not known if p can be a non-denumerable set.

B. A. Amirà (Jerusalem)

5716:

Gutmann, Marcian. Sur quelques formules intégrales classiques de la théorie des fonctions analytiques. Bul. Inst. Politehn. București **20** (1958), no. 1, 17–22. (Russian, English and German summaries)

According to a reference, this paper is a communication delivered to a group of teachers. The author "proves" some well-known theorems in elementary complex variable theory in just that plausible manner which one uses when teaching an engineering class which has little interest in strict mathematics. An outline of one example is as follows: Define

$$I(\lambda) = \oint \lambda f[a+\lambda(z-a)]dz,$$

where $f(z)$ is analytic. Differentiate with regard to λ (formally). Using integration by parts it is easy to show that $I'(\lambda)=0$ for all λ. But $I(0)=0$. Then put $\lambda=1$ to obtain $\oint f(z)dz=0$. *J. L. Griffith* (Kensington)

5717:

Chou, Hsin-ti. On the uniqueness theorem of analytic functions and its applications. Acta Math. Sinica **9** (1959), 114–120. (Chinese. English summary)

The author proves the following uniqueness theorem. Let $f(z)$ be an analytic function regular in a domain D, which is the interior region bounded by a certain rectifiable Jordan curve Γ, composed of two curves Γ' and Γ'' with a and b as their two end points. Let $f(z)$ be continuous on Γ' $(a, b \in \Gamma')$. If there exists a sequence of simple rectifiable curves (λ_n), belonging to D except for their end points z_n' and z_n'', which are points of Γ'', and converging to Γ'' as $n\to\infty$, such that $\lim_{n\to\infty} \int_{\lambda_n} |f(z)|\,|dz|=0$, then $f(z)\equiv 0$ in D.

Using this theorem, the author gives an application concerning the uniform convergence of the sequence $\{f_n(z)\}$ in D. *Fu Cheng Hsiang* (Taipeh)

5718:

Portmann, Walter O. A derivative for Hausdorff-analytic functions. Proc. Amer. Math. Soc. **10** (1959), 101–105.

If the hypercomplex function $f(z)$ is H-analytic (i.e., analytic in the sense of Hausdorff [Leipziger Berichte **52** (1900), 43–61]) in a domain D, then $df(z)=\sum_{k=1}^{k=n} u_k dz v_k$. The author defines $df(z)/dz=\sum_{k=1}^{k=n} u_k v_k$ and proves the usual theorems for sums and products. He applies the theory to the algebra M of square matrices over the complex field F. He proves that if the components of $f(Z)$ are analytic functions in F of the z_{ij}, then $f(Z)$ is H-analytic in M and $df(Z)/dZ$ is the sum of the partial derivatives of $f(Z)$ with respect to the main diagonal elements of Z. He also proves: If $f(z)$ is a scalar function which is analytic at the eigenvalues of Z_0 in M, then the corresponding matric function $f(Z)$ is H-analytic in a neighborhood of Z_0 and $df(Z_0)/dZ=f'(Z_0)$.

J. A. Ward (Holloman Air Force Base, N.M.)

5719:

Blambert, Maurice. Sur la transformation de Mellin et les fonctions à dominante angulaire algébrico-logarithmique en un point. Ann. Inst. Fourier. Grenoble **8** (1958), 367–407.

Der Verf. setzt seine Untersuchungen über die Hadamard-Mandelbrojt-Komposition $H_k(f, \varphi|s)=\sum a_{\mu_n}{}^{(k)} b_n e^{-\mu_n s}$ zweier allgemeiner Dirichletreihen $f(s)=\sum a_n e^{-\lambda_n s}$ und $\varphi(s)=\sum b_n e^{-\mu_n s}$ fort [vgl. Acta Math. **89** (1953), 217–242; Rend. Circ. Mat. Palermo (2) **3** (1954), 214–243; Ann. Sci. École Norm. Sup. (3) **72** (1955), 199–235; Publ. Sci. Univ. Alger. Sér A **2** (1955), 251–272; MR **15**, 206; **16**, 583; **17**, 722; **19**, 134]. Ein neuer Typ von Singularitäten wird eingeführt. Eine im Winkelraum Σ: $|\arg(z-z_0)-\theta_0|<\eta/2$

reguläre Funktion $\varphi(z)$ besitzt in Σ am Punkt z_0 eine "algebraisch-logarithmische Winkel-Dominante", wenn $\varphi=\varphi_0+\psi$ gilt mit einer Funktion ψ, die an z_0 eine algebraisch-logarithmische Singularität hat, während φ_0 eine gegenüber ψ kleine Korrektur ist: $|\varphi_0(z)|=O(|z-z_0|^{-\infty})$ ($|z-z_0|\to 0$ in Σ). Die Hauptsätze der Arbeit in Kap. IV schließen von gewissen Annahmen über die Singularitäten von f, φ auf die Singularitäten der Komposition $H_k(f, \varphi|s)$. Als Beweishilfsmittel wird die Mellin-Transformation

$$\varphi^*(s) = (\Gamma(s))^{-1} \int_0^\infty \varphi(z)(z-z_0)^{s-1}dz, \quad z=z_0+\rho e^{i\theta_0},$$

einer im Winkelraum regulären Funktion $\varphi(z)$ herangezogen, und die Kap. II und III setzen die Singularitäten von φ und φ^* in Zusammenhang. Für den Fall, daß z_0 regulärer Punkt von φ ist, erweist sich φ^* als ganze Funktion, die in jedem Halbstreifen $|\tau|<T$, $\sigma<\sigma_0$ von der Ritt-Ordnung Null ist. Die genauen Sätze sind zu kompliziert für die Erwähnung in einem Referat. *D. Gaier* (Giessen)

5720:

Gallie, T. M., Jr. Mandelbrojt's inequality and Dirichlet series with complex exponents. Trans. Amer. Math. Soc. **90** (1959), 57–72.

L'auteur considère une série de Dirichlet

$$(1) \qquad \sum_{n=1}^{\infty} c_n \exp(-\lambda_n z)$$

à exposants complexes λ_n satisfaisant $|\lambda_n|\nearrow$, $\sup(n/|\lambda_n|)<\infty$. Soit D le domaine de convergence de (1), $g(z)$ une fonction entière s'annulant sur $\{\lambda_n\}$, et C' un compact convenablement lié à g. Supposant que la somme f de (1) est analytiquement prolongeable le long des translatés $z+C'$ quand z parcourt un arc (z_1, z_2) tel que $z_1+C'\subset D$, il démontre l'inégalité $|c_n|\leq km|\exp(\lambda_n z_2)/g'(\lambda_n)|$ ($m=\sup|f(z)|$ quand $z\in z_2+C'$, $k=k(g, C')$), qui généralise le théorème 3.7 de Mandelbrojt [*Séries adhérentes*, Gauthier-Villars, Paris, 1952; MR **14**, 542]. Il en tire, comme Mandelbrojt, des théorèmes sur la convergence et la localisation des singularités de (1), qui recouvrent partiellement les résultats de Leont'ev [Trudy Mat. Inst. Steklov **39** (1951); MR **14**, 1074] et du référent [Ann. Inst. Fourier, Grenoble **5** (1953–54), 39–130; MR **17**, 732]. D'une autre inégalité, beaucoup plus facile, l'auteur tire des résultats sur les zéros des sommes partielles et sur l'ultraconvergence de (1). *J. P. Kahane* (Montpellier)

5721:

Koiter, W. T. Some general theorems on doubly-periodic and quasi-periodic functions. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 120–128.

Der Verfasser untersucht Randwertprobleme in einer Ebene mit doppeltperiodisch angeordneten Löchern, indem er im Cauchy-Integral [vgl. N. I. Muskhelišvili, *Singulyarnye integral'nye uravneniya*, OGIZ, Moscow-Leningrad, 1946; Übersetzung von J. R. M. Radok, Noordhoff, Groningen, 1953; MR **8**, 586; **15**, 434] den Kern $(t-z)^{-1}$ durch die Weierstraßsche Zetafunktion ersetzt. Er gibt die Bedingungen für periodische Lösungen an und Reihenentwicklungen für den Fall kreisförmiger Berandungen. *F. Stallmann* (Braunschweig)

5722:

Boyarskii, B. V. The Riemann-Hilbert problem for a holomorphic vector. Dokl. Akad. Nauk SSSR **126** (1959), 695–698. (Russian)

The problem consists of determining a holomorphic (or meromorphic) vector $\varphi=(\varphi_1, \cdots, \varphi_n)$ in a plane domain D with boundary L satisfying the boundary condition $\mathrm{Re}[A(t)\varphi(t)]=f(t)$ on L, where $A=\{a_{ik}(t)\}$ is a given matrix function of $t\in L$ and f is a given (real) vector. The author denotes by Ω the set of matrix functions $A(t)$, $\det A(t)\neq 0$, which are Hölder continuous on L, by Ψ the set of non-singular matrix functions of order n which are holomorphic in D and Hölder continuous in $D\cup L$. The number $\kappa=\kappa(A)=(2\pi)^{-1}\Delta_L \arg\det A(t)$ is called the index of a matrix $A\in\Omega$. Two matrix functions $A_1\in\Omega$, $A_2\in\Omega$ are called equivalent with respect to the problem ($A_1 \overset{R}{\sim} A_2$) if $A_1(t)=S(t)A_2(t)\psi(t)$, where $\psi(t)$ are the boundary values on L of the matrix $\psi(t)\in\Psi$, and where $S(t)$ is a real matrix. The 2×2 matrices

$$\begin{pmatrix} z^{k+1} & iz^k \\ 0 & z^k \end{pmatrix} = z^k\begin{pmatrix} z & i \\ 0 & 1 \end{pmatrix},$$

or briefly (z^{k+1}, z^k) or $(k+1, k)$, are called elementary cells with partial index k, and Λ denotes the set of quasidiagonal matrices $\lambda(t)$ of the form $\{Q_i(t)\}$, where $Q_i(t)$ is either $t^{\kappa_i'}$ or $(t^{\kappa_i''+1}, t^{\kappa_i''})$, the numbers κ_i' and κ_i'' being called respectively the free and bound partial indices of $\lambda(t)$; $\lambda(t)$ is called normed if $\kappa_i\geq\kappa_j$ for $i\leq j$ ($\{\kappa_i\}$, $i=1, \cdots, n$, is the sequence of exponents down the diagonal).

It is asserted that in every equivalence class defined by the relation $\overset{R}{\sim}$, there exists a matrix $\lambda(t)\in\Lambda$ which is unique if $\lambda(t)$ is normed, and that the partial indices of a matrix $A(t)\in\Omega$ are related to the index κ of $A(t)$ by the formula $\kappa=\sum_{i=1}^n \kappa_i$.

The main object of the paper is to announce the result that the number of solutions of the homogeneous Riemann-Hilbert problem which are linearly independent over the real field is given by

$$l(A) = \sum_{j=1}^{r} 2|\kappa_{k_j}|+r+\sum_{j=1}^{p} 4|\kappa_{l_j}|,$$

where the first sum is taken over all free partial indices $\kappa_{k_j}\leq 0$, and the second over all cells with negative index. The discussion is limited to the case that D is simply connected. *A. J. Lohwater* (Houston, Tex.)

5723:

Foos, Sister Barbara Ann. The values of certain sets of modules. Duke Math. J. **26** (1959), 467–484.

Es werden folgende konforme Grössen betrachtet: Der Modul M eines zweifachzusammenhängenden ebenen Gebietes R; der reduzierte Modul M' (bezüglich O) eines einfachzusammenhängenden Gebietes G', das O enthält, d.i. $M'=(2\pi)^{-1}\log r$, wenn r den inneren Radius von G' bezüglich O bezeichnet; und der reduzierte Modul M'' (bezüglich ∞) eines einfachzusammenhängenden Gebietes G'', das ∞ enthält.

Es wird die Menge der Zahlenpaare (M', M'') vollständig charakterisiert, wenn G' und G'' fremd zueinander und zu $z=1$ sind; ebenso die Menge der Paare (M', M), wenn G' und R im Kreis $|z|<1$ gelegen sind, zueinander und zu einem festen Punkt $z=\alpha$ ($0<\alpha<1$) fremd sind und G' und α durch R von der Peripherie $|z|=1$ getrennt werden;

ebenso die Menge der Paare (M', M, M''), wenn G', R und G'' fremd sind zueinander und zu zwei festen Punkten $z=\alpha$, $z=\beta$ ($0<\alpha<\beta<\infty$), sowie G' und α durch R von β und G'' getrennt werden. Verwendet werden Methoden der quadratischen Differentiale und der extremalen Metrik (Extremallängen).
A. Pfluger (Zürich)

5724:

Jenkins, James A. On a type problem. Canad. J. Math. **11** (1959), 427–431.

Suppose that the identification on two edges of the half-strip $x>a$, $0<y<b$ in the $z=x+iy$-plane defined by the mapping $T(x)=f(x)+ib$ $(x\geqq a)$ determines a Riemann surface $\mathfrak{R}$. Here $f(x)$ $(x\geqq a)$ is an increasing function with $f(a)>a$ which is absolutely continuous together with its inverse. As an improvement of Volkovyskiĭ's result the author shows that under some other restrictions on $f(x)$ the surface $\mathfrak{R}$ is of hyperbolic type if $f_n(a)\to\infty$ as $n\to\infty$ and $\sum f_n'(x)^{-1}$ converges on a set of positive measure on $a\leqq x<f(a)$ where $f_n(x)$ denotes the nth iterate of $f(x)$.
Y. Komatu (Tokyo)

5725:

Matsumoto, Kikuji. Remarks on some Riemann surfaces. Proc. Japan Acad. **34** (1958), 672–675.

Let f be a conformal mapping of a Riemann surface R_1 into another, R_2. The author calls f of type-Bl at $q\in R_2$ if, in essence, q belongs to a Jordan region Ω such that, for each component Δ of $f^{-1}(\Omega)$, the restriction to Δ of f is of type-Bl in the sense of M. Heins [Ann. of Math. (2) **61** (1955), 440–473; MR **16**, 1011]. If f is of type-Bl at every point of R_2, then f is said to be locally of type-Bl. It is shown that f has this property if and only if, for any compact subregion Ω of R_2, each component of $f^{-1}(\Omega)$ belongs to the class SO_{HB} introduced by T. Kuroda [Nagoya Math. J. **6** (1953), 77–84; MR **15**, 519].

The author then announces, without proofs, several consequences of this theorem and of a theorem of Z. Kuramochi [Osaka Math. J. **7** (1955), 109–127; MR **17**, 26]. These applications concern covering properties of inverse functions of meromorphic functions, and the classes O_{HB_n} and O_{HD_n} of C. Constantinescu and A. Cornea [Nagoya Math. J. **13** (1958), 169–233; MR **20** #3273].
L. Sario (Los Angeles, Calif.)

5726:

Nakai, Mitsuru. A function algebra on Riemann surfaces. Nagoya Math. J. **15** (1959), 1–7.

Let R_1 and R_2 be two compact Riemann surfaces, and $M(R_1)$ and $M(R_2)$ the rings on them of continuous functions with a finite Dirichlet integral. Let $\|f\|=\sup|f(p)|+\{D(f)\}^{1/2}$. Then the author shows that R_1 and R_2 are conformally equivalent if and only if the rings $M(R_1)$ and $M(R_2)$ are isometrically isomorphic.
H. L. Royden (Stanford, Calif.)

5727:

Heins, Maurice. On the principle of harmonic measure. Comment. Math. Helv. **33** (1959), 47–58.

Utilisation d'une mesure harmonique généralisée pour une extension du principe de la mesure harmonique de Nevanlinna. Une fonction harmonique u sur une surface de Riemann F (avec $0\leqq u\leqq 1$) est une mesure harmonique généralisée si la plus grande minorante harmonique de u et $1-u$ est nulle. Soient F et G deux surfaces de Riemann appliquées conformément l'une dans l'autre par $\varphi: F\to G$. On obtient une inégalité simple entre les mesures harmoniques généralisées u et v respectivement sur F et G. Entre autres résultats importants on trouve d'intéressants propriétés de minimalité pour la classe (P) des fonctions super-harmoniques sur G vérifiant $u\leqq P\circ\varphi$ et la classe (v) des mesures harmoniques généralisées sur G, vérifiant $u\leqq v\circ\varphi$ (u est une mesure harmonique généralisée sur F).
L. Fourès (Marseille)

5728:

Gel'fer, S. A. The maximum of the conformal radius of the fundamental region of a group of linear fractional transformations. Dokl. Akad. Nauk SSSR **126** (1959), 463–466. (Russian)

Let $\{T\}$ be a discrete group of fractional-linear transformations and $\{D\}$ a family of simply-connected domains D in the w-plane with the following properties: (1) D does not contain points congruent relatively to the group $\{T\}$; (2) D does not contain a system of finite points $a_1,\cdots,a_m$ and ∞, nor points congruent to them by the group $\{T\}$; (3) D contains a given point c_0, different from the fix-points and cyclic points of the transformations in $\{T\}$. The problem is to find the domain $D\in\{D\}$ which has the maximum conformal radius relatively to c_0. The author deals in this paper with groups for which simple automorphic functions exist. He solved in another paper a similar problem for doubly-periodic groups [S. A. Gel'fer, Mat. Sb. (N.S.) **44** (**86**) (1958), 213–224; Dokl. Akad. Nauk SSSR **114** (1957), 241–244; MR **20** #1782, #5894].

Denote by $S_a(T)$ the class of functions $w=f(z)=\sum_0^\infty c_nz^n$ regular in $|z|<1$ which provide a univalent mapping of $|z|<1$ on the domains of $\{D\}$. The problem then is to find the max $|f'(0)|$ in the class $S_a(T)$.

The following theorem is proved. If a function $f(z)\in S_a(T)$ gives a maximum value to the functional $|f'(0)|$, then $f(z)$ maps $|z|<1$ on a domain D with the following properties: (1) D, containing the point c_0, is the fundamental region S_0 of the group $\{T\}$ with slits. Its boundary consists of a finite number of analytic arcs pairwise congruent relatively to $\{T\}$ and of piece-wise analytic slits going from the boundary of S_0 to the points $a_1,\cdots,a_m$ or to points congruent to them. (2) To each pair of congruent arcs of the boundary of S_0 and to the simple arcs of the slits correspond on $|z|=1$, through the mapping $w=f(z)$, two arcs of equal length. (3) D is unique for the given group.

In some special cases this theorem helps to find the extremal domain. Thus, for the modular group $\{T\}$: $w'=(aw+b)/(cw+d)$, $ad-bc=1$, a, b, c, d real integers, and for the subgroup of $\{T\}$ in which b and c are even integers, as well as for groups of which the fundamental regions are images of closed Riemann surfaces of genus $g\geqq 2$, explicit expressions for the upper bound of $|f'(0)|$ are given.
B. A. Amirà (Jerusalem)

5729:

Shankar, Hari. On Lindelöf's theorems on entire functions. J. Indian Math. Soc. (N.S.) **22** (1958), 137–147.

If $L(x)$ is continuous for all large enough values of x and such that $L(ax)\sim L(x)$ for every fixed $a>0$, and if $f(z)$ is an entire function of order ρ, then $f(z)$ is defined to be of

maximal (max) L-type, mean L-type or minimal (min) L-type of the order ρ according to the

$$\limsup_{r\to\infty} \log\left[\max_{|z|=r} |f(z)|\right]\Big/[r^\rho L(r)]$$

being infinite, finite and non-zero, or zero, respectively. Using this definition the author generalizes three theorems of Lindelöf [see R. P. Boas: *Entire functions*, New York, 1954; MR **16**, 914; theorems 2.9.5, 2.10.1, and 2.10.3] relating the ordinary type with the asymptotic behavior of the function $n(r)$ = number of zeros of $f(z)$ in the circle $|z|\leq r$. For example: If ρ is not an integer, then $f(z)$ is of mean L-type if and only if $n(r)=O[r^\rho L(r)]$ and of min L-type if and only if $n(r)=o[r^\rho L(r)]$.

A. G. Azpeitia (Amherst, Mass.)

5730:

Srivastav, R. P. On the mean value of integral functions and their derivatives. Riv. Mat. Univ. Parma **8** (1957), 361–369.

Let $f(z)$ be an entire function of order ρ and lower order λ. The mean value of $f(z)$ for $|z|=r$ is

$$I(r) = (1/2\pi)\int_0^{2\pi} |f(re^{i\theta})|d\theta$$

and its logarithmic mean value for $|z|=r$ is

$$L(r) = (1/2\pi)\int_0^{2\pi} \log|f(re^{i\theta})|d\theta.$$

The author obtains a number of properties of $I(r)$, $L(r)$ and the mean value $I^{(s)}(r)$ $(s=1, 2, \cdots)$ of the sth derivative of $f(z)$. For example: (i) If $M(r)=\max_{|z|=r}|f(z)|$, then

$$I(r) \leq M(r) \leq [(R+r)/(R-r)]I(R)$$

for $R>r>0$;

(ii) $$\limsup_{r=\infty} \log\{r[I^{(s)}(r)/I(r)]^{1/s}\}/\log r = \rho;$$

(iii) if $n(r)$ is the number of zeros of $f(z)$ in $|z|\leq r$ and

$$T = \limsup_{r=\infty} L(r)/r^\rho, \qquad t = \liminf_{r=\infty} L(r)/r^\rho,$$

$$\gamma = \limsup_{r=\infty} n(r)/r^\rho, \qquad \delta = \liminf_{r=\infty} n(r)/r^\rho,$$

then $\rho\leq \gamma e^{\delta/\gamma}/e\leq\rho T\leq\gamma$ and $\delta\leq\rho t\leq\delta[1+\log(\gamma/\delta)]\leq\gamma$.

A. G. Azpeitia (Amherst, Mass.)

5731:

Valiron, Georges. Fonctions entières d'ordre fini et fonctions méromorphes. V. Enseignement Math. (2) **5** (1959), 1–28.

Continuation of Enseignement Math. (2) **4** (1958), 1–18, 124–156, 157–177, 229–271 [MR **21** #2048]. This is the final part of the author's lecture course and deals with some parts of the theory of meromorphic functions in the spirit of Nevanlinna and Ahlfors.

R. C. Buck (Princeton, N.J.)

5732:

Lehto, Olli. The spherical derivative of meromorphic functions in the neighbourhood of an isolated singularity. Comment. Math. Helv. **33** (1959), 196–205.

In a series of earlier papers [Acta Math. **97** (1957), 47–65; Ann. Acad. Sci. Fenn. Ser. A.I, no. 240 (1957); MR **19**, 403, 404], the author and Virtanen have introduced the spherical derivative $\rho(f(z))=|f'(z)|[1+|f(z)|^2]^{-1}$ as a natural measure of the growth of a meromorphic function in the neighborhood of a singularity $z=a$, and have shown the existence of an absolute constant $k\geq 1/2$ such that $\limsup_{z\to a}|z-a|\rho(f(z))\geq k$. It is shown first that equality holds for the Weierstrass products $\prod_\nu (z-a-a_\nu)/(z-a+a_\nu)$, where the numbers a_ν satisfy the condition $|a_{\nu+1}|=o(|a_\nu|)$. It is then shown that, for an entire function

$$f(z)=\prod(1-z/a_\nu),$$

if the zeros satisfy the condition $|a_{\nu+1}/a_\nu|\geq q>1$, then

$$\liminf_{\nu\to\infty} \rho(f(a_\nu))|a_\nu| \exp[-N(|a_\nu|, 0)]>0,$$

which shows that the spherical derivative can grow rapidly for an entire function of slow growth in terms of the counting function $N(r, a)$.

It is shown that if $f(z)$ is of sufficiently rapid spherical growth (in a sense to be defined) in the neighborhood of an isolated singularity $z=a$, then Picard's theorem holds for $f(z)$ in the union of any infinite subsequence of the circles $|z-z_\nu|<\varepsilon h(|z_\nu-a|)$ for each $\varepsilon>0$, where $z_\nu\to a$ and $h(r)$ is a positive function of r with $h(r)=O(r)$ as $r\to 0$, and such that $h(|z_\nu-a|)\rho(f(z_\nu))\to\infty$.

Certain necessary conditions and sufficient conditions are given in terms of the spherical derivative for the existence of a Julia radius of a function meromorphic in $|z|<1$.

A. J. Lohwater (Houston, Tex.)

5733:

Dolženko, E. P. Boundary value theorems on the uniqueness and behaviour of analytic functions near the boundary. Dokl. Akad. Nauk SSSR **129** (1959), 23–26. (Russian)

The author deals with various questions related to the uniqueness theorem of Lusin and Privaloff [Ann. Sci. École Norm. Sup. **42** (1925), 143–191]. It is shown first that for every r $(0<r<1)$ and for every complex number a there exists in $|z|<1$ an analytic function such that (1) for every $\zeta\in C_1: |z|=1$ there exists a rectilinear segment L^ζ joining ζ with some point z_ζ on $|z|=r$ such that $f(z)\to 0$ as $z\to\zeta$ on L^ζ, the limit being uniform for $\zeta\in C_1$ in the sense that $f(z)\to 0$ as $|z|\to 1$ with $z\in\bigcup_{\zeta\in C_1} L^\zeta$; (2) the function $\varphi(\zeta)=z_\zeta$ is continuous on the set

$$\{\zeta = e^{i\vartheta}: \vartheta \neq \frac{\pi}{3}\frac{m}{2^n};\ m, n = 0, \pm 1, \pm 2, \cdots\};$$

and (3) $f(z)\neq a$ in $|z|<1$.

The remainder of the paper is devoted to some rather obvious modifications of uniqueness theorems of Collingwood [Acta Math. **91** (1954), 165–185; Ann. Acad. Sci. Fenn. Ser. A.I, no. 250/6 (1958); MR **16**, 460; **20** #2451] and of Bagemihl and Seidel [Nagoya Math. J. **9** (1955), 79–85; MR **17**, 471].

A. J. Lohwater (Houston, Tex.)

5734:

Sakaguchi, Kôichi. On a certain univalent mapping. J. Math. Soc. Japan **11** (1959), 72–75.

The author shows that the inequality

$$\mathrm{Re}[zf'(z)/\{f(z)-f(-z)\}]>0$$

characterizes a class of normalized univalent functions; he terms them "starlike with respect to symmetrical

points" and shows that they are close-to-convex. He also introduces certain generalizations of this class. Of interest is a lemma asserting that, if $zf'(z)$ is univalent and close-to-convex for $|z|<1$, then so also is $f(z)$.

W. Kaplan (Ann Arbor, Mich.)

5735a:

Mocanu, Petru T. Une généralisation du théorème de la contraction dans la classe S de fonctions univalentes. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. 8 (1957), 303–312. (Romanian. Russian and French summaries)

5735b:

Mocanu, Petru T. Sur une généralisation du théorème de contraction dans la classe des fonctions univalentes. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. 9 (1958), 149–159. (Romanian. Russian and French summaries)

Let $f(z)=z+a_2z^2+\cdots$ be analytic and univalent in the unit disk. Let $\rho=\rho(\theta)$ define a closed Jordan curve lying in the unit disk and surrounding the origin. The author shows that the image of the domain bounded by this curve contains a certain universal star-shaped domain and is contained in another one; when $\rho(\theta)$ is constant, the result reduces to the classical distortion theorem $r(1+r)^{-2}\leq|f(z)|\leq r(1-r)^{-2}$. The extremal domains are the stars of domains whose boundaries are defined by rather complicated parametric equations. In the second paper the author determines some properties of these extremal domains, discusses the case when the original curve passes through the origin, and analyzes some special cases. {The author's paper Gaz. Mat. Fiz. Ser. A **10** (**63**) (1958), 473–477 [MR **20** #5289] falls logically between these two.}

R. P. Boas, Jr. (Evanston, Ill.)

5736:

Royster, W. C. Coefficient problems for functions regular in an ellipse. Duke Math. J. **26** (1959), 361–371.

The coefficient problem treated in this paper concerns the expansion $f(z)=\sum_{n=0}^\infty a_nT_n(z)$, where the $T_n(z)$ are the Tchebychef polynomials, of an analytic function regular in an ellipse E of foci ±1. The functions considered are univalent in E and map E onto domains subject to certain restrictions of a geometric nature, such as symmetry with respect to the real axis, starlikeness, etc. For all these classes, coefficient inequalities are obtained. Example: If $a_0=0$, $a_1=1$, $f(z)$ is univalent in an ellipse of half-axes a and b, and the a_n are real, then

$$|a_n| \leq n[R-R^{-1}][R^n-R^{-n}]^{-1},$$

where $R=a+b$. The inequality is sharp for all n.

Z. Nehari (Pittsburgh, Pa.)

5737:

Clunie, J. On meromorphic schlicht functions. J. London Math. Soc. **34** (1959), 215–216.

The author gives a simple and ingenious proof of the following result: If $a_1, a_2, \cdots$ are the coefficients of a function $f(z)=z^{-1}+a_1z+a_2z^2+\cdots$ which is schlicht in $|z|<1$ and maps $|z|<1$ onto the complement of a point set starlike with respect to the origin, then (*) $|a_n|<2(n+1)^{-1}$ unless $f(z)$ is of the form $f(z)=z^{-1}(1+\lambda z^{n+1})^{2/(n+1)}$, $|\lambda|=1$. For $1\leq n\leq 6$, (*) had been proved by the reviewer and Netanyahu [Proc. Amer. Math. Soc. 8 (1957), 15–23; MR 18, 648].

Z. Nehari (Pittsburgh, Pa.)

5738:

Havin, V. P. The rate of growth of functions of the H_p class and the multiplication of integrals of the Cauchy-Stieltjes type. Dokl. Akad. Nauk SSSR **127** (1959), 757–759. (Russian)

Let $\lambda(r)$ be a positive, monotone increasing, continuous function of r on $[0, 1)$ and denote by X_λ the set of all functions f regular in $|z|<1$ and satisfying the condition $M(r,f)=O(\lambda(r))$, where $M(r,f)=\max_{0\leq\theta\leq2\pi}|f(re^{i\theta})|$. The author shows with reference to the class of functions H_p $(p>0)$, that if $H_p\cap X_\lambda\subset\bigcup_{\varepsilon>0}H_{p+\varepsilon}$, then $\sup_{0<r<1}\lambda(r)<\infty$.

The second part of this brief note states without proof two theorems concerning functions of class K, i.e., functions f, regular in the unit circle and representable there by integrals of Cauchy-Stieltjes type $\left(f(z)=\int_0^{2\pi}\frac{dg(\theta)}{e^{i\theta}-z}\right.$, g a function of bounded variation). (Necessary and sufficient conditions for $f\in K$ were given previously by the author [Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **13** (1958), no. 1, 66–79; MR **20** #1762].) We quote the second of these theorems: If φ is regular in the unit circle and is such that

$$\sum_{n=2}^\infty \frac{1}{n!}|\varphi^{(n)}(0)|\log n<\infty,$$

then $\varphi\psi\in K$ if $\psi\in K$.

J. F. Heyda (Cincinnati, Ohio)

5739:

Komatu, Yûsaku. On the range of analytic functions with positive real part. Kōdai Math. Sem. Rep. **10** (1958), 145–160.

Let R_0 denote the class of analytic functions which are regular and of positive real part in $|z|<1$ with $\Phi(0)=1$. Let R_q denote the class of analytic functions which are single-valued, regular and of positive real part in $(0<)q<|z|<1$ and normalized by $\operatorname{Re}\Phi(z)=1$ on $|z|=q$ and $\int_{-\pi}^{\pi}\operatorname{Im}\Phi(qe^{i\theta})d\theta=0$. Let $\hat{R}_q$ denote the class of analytic functions which are single-valued, regular and of positive real part in $(0<)$ $q<|z|<1$ and normalized by $\int_{-\pi}^{\pi}\Phi(re^{i\theta})d\theta=1$. For fixed z, the range-set $\Omega_z(\mathscr{F})$ of a family of functions $\mathscr{F}$ is the set of values $\Phi(z)$ for all $\Phi\in\mathscr{F}$. For $|z|=r$, $\Omega_z(R_0)$ is shown to be the closed circular disk $|(w-1)/(w+1)|\leq r$ $(0\leq r<1)$. For $|z|=r$ $(q\leq r<1)$, $\Omega_z(R_q)$ is the closed convex set bounded by the image curve of $|z|=r$ by the mapping $w=(2/i)(\zeta(i\log z)-(\eta_1 i/\pi)\log z)$ where ζ is the Weierstrass ζ-function for the periods $2\omega_1=2\pi$ and $2\omega_2=-2i\log q$ and $\eta_1=2\zeta(\omega_1)$. The range set $\Omega_z(\hat{R}_q)$ is also described as well as the characterizations of the unique functions corresponding to boundary points of the range sets for all three classes of functions. The author then generalizes these results to the range sets of classes obtained from R_0 or R_q by applying certain linear operators to them.

G. Springer (Lawrence, Kans.)

5740:

Komatu, Yûsaku. On convolution of Laurent series. Proc. Japan Acad. **34** (1958), 649–652.

Let

$$f(z) = 1+2\sum_{n=-\infty}^{\infty}{}' a_n(1-q^{2n})^{-1}z^n,$$

$$g(z) = 1+2\sum_{n=-\infty}^{\infty}{}' b_n(1-q^{2n})^{-1}z^n$$

be regular and of positive real part for $0<q<|z|<1$. If, in addition, both $\operatorname{Re}\{f(z)\}$ and $\operatorname{Re}\{g(z)\}$ are constant for $|z|=q$, the author has proved [Kōdai Math. Sem. Rep. **10** (1958), 141–144; MR **21** #714] that the function

$$h(z)=1+2\sum_{n=-\infty}^{\infty}{}' a_n b_n(1-q^{2n})^{-1}z^n$$

also satisfies the conditions $\operatorname{Re}\{h(z)\}>0$ $(q<|z|<1)$, $\operatorname{Re}\{h(z)\}=\text{const}$ $(|z|=q)$. In the present paper the author discusses the possibility of extending this result to functions which are not assumed to have a constant real part on one of the boundaries of the ring. Although the result does not necessarily hold under these weakened assumptions—an example is exhibited which shows that the composite series does not even have to converge—a modified form of the theorem is shown to be valid in the general case.

Z. Nehari (Pittsburgh, Pa.)

5741:

Kennedy, P. B. A property of bounded regular functions. Proc. Roy. Irish Acad. Sect. A **60** (1959), 7–14.

Let $f(z)$ be regular in $|z|<1$ and define

$$m(r,f)=(2\pi)^{-1}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta,\quad 0<r<1.$$

Let R_f be the set of θ in $(-\pi,\pi)$ for which $e^{i\theta}$ is a regular point of $f(z)$, and let S_f be the set of θ in $(-\pi,\pi)$ for which $e^{i\theta}$ is a singular point of $f(z)$. Denote by $2\pi\mu_f$ the measure of S_f and define

$$\lambda_f=\limsup_{r\to 1} m(r,f')\left(\log\frac{1}{1-r}\right)^{-1}.$$

The author shows that if $f(z)$ is bounded, then $\lambda_f\leqq\mu_f\leqq 1$. In particular, if $f(z)$ is bounded and $\lambda_f=1$, then $f(z)$ is non-continuable across $|z|=1$. This theorem is best-possible for any given μ_f in the closed interval $(0,1)$.

M. S. Robertson (New Brunswick, N.J.)

5742:

Kennedy, P. B. On a theorem of Hayman concerning quasi-bounded functions. Canad. J. Math. **11** (1959), 593–600.

Let $f(z)$ be regular in $|z|<1$. If

$$m(r,f)=(2\pi)^{-1}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta$$

is bounded for $0<r<1$, $f(z)$ is called quasi-bounded. Hayman [Rend. Circ. Mat. Palermo, Ser. (2) **2** (1953), 346–392; MR **16**, 122] has given sufficient conditions that $f'(z)$ be also quasi-bounded.

A domain D is said to properly contain a sequence of non-overlapping open arcs

$$(1)\qquad z=e^{i\theta},\quad \theta_n<\theta<\theta_n+\delta_n\quad (n=1,2,3,\cdots),$$

if the distance $d(\theta)$ from $e^{i\theta}$ to the boundary of D satisfies

$$d(\theta)>A\{(\theta-\theta_n)(\theta_n+\delta_n-\theta)\}^B,\qquad (A>0,\ B>0\text{ constants}).$$

Hayman showed that if D contains $|z|<1$ and properly contains the sequence of arcs (1) such that

$$(2)\qquad \sum_1^\infty\delta_n=2\pi,\qquad \sum_1^\infty\delta_n\log\frac{1}{\delta_n}<\infty,$$

and if $f(z)$ is regular and bounded in D, then $f'(z)$ is quasi-bounded in $|z|<1$. The author shows that if the inequality in (2) is replaced by

$$(3)\qquad \limsup_{n\to\infty}\left(\sum_n^\infty\delta_j\right)\log\frac{1}{\delta_n}=\infty,$$

then $m(r,f')\to\infty$ as $r\to 1$. In particular, the inequality in (2) cannot in general be replaced by

$$(4)\qquad \sum_1^\infty\delta_n\left(\log\frac{1}{\delta_n}\right)^\alpha<\infty,\quad\text{for any }\alpha<1.$$

M. S. Robertson (New Brunswick, N.J.)

5743:

Baker, I. N. Fixpoints and iterates of entire functions. Math. Z. **71** (1959), 146–153.

Let f be an entire transcendental function. For n a natural number define $f_1=f$, $f_n=f_{n-1}(f)$. Then f_n is called the nth natural iterate of f. A fixpoint of order n of f is a complex number ξ such that $f_n(\xi)=\xi$. A fixpoint is exactly of order n if $f_k(\xi)=\xi$ holds for $n=k$ but for no lower integer. The value $f_n'(\xi)$ is called the multiplier of ξ.

Previous results by Fatou [Acta Math. **47** (1926), 337–370] and Rosenbloom [C. R. Acad. Sci. Paris **227** (1948), 382–383; Medd. Lunds Univ. Mat. Sem. Suppl. Band (1952), 186–192; MR **10**, 187; **14**, 546] established that every f has an infinite number of fixpoints and that it has an infinite number of fixpoints of every order $n>1$. Even the second result does not insure that there are any fixpoints of exact order n, for a given $n>1$. Employing results proved in an earlier paper [Math. Z. **69** (1958), 121–163; MR **20** #4000] the author obtains a relation between the number of fixpoints ξ with $|\xi|<r$ of order n and the maximum modulus of f_n for $|z|<r$. The relation holds only for functions f of order less than 1/2. From this result the following theorem is derived: If the order of f is less than 1/2 and if no multiplier of a fixpoint of f of order 1 is 1 or a pth root of unity for a certain prime p, then f has infinitely many fixpoints of the exact order p.

W. J. Thron (Boulder, Colo.)

5744:

Szekeres, G. Regular iteration of real and complex functions. Acta Math. **100** (1958), 203–258.

This article is devoted to a study of solutions x of the Schroeder functional equation $\chi(f)=\alpha\chi$, for given f and α, and the closely related problem of determining the iterates of f. A family of functions (f_σ), where σ may be any real or in certain cases any complex number, is called a system of iterates of f provided it satisfies the two conditions: $f_1=f$, $f_\sigma(f_\tau)=f_{\sigma+\tau}$. In general neither the solutions of Schroeder's equation nor the families of iterates of a given function f are unique. With suitable restrictions on f, x, and (f_σ), however, the following or similar connections exist: $f_\sigma=\chi^{-1}(\alpha^\sigma\chi)$, $\chi=\lim_{n\to\infty}\alpha^{-n}f_n$. The author establishes sufficient conditions for the existence of χ and families (f_σ) and for connections between the two. Throughout the article f is assumed to be real for $x\geqq 0$ and to have a fixed point at $x=0$. While heretofore the problem had been mainly studied in the case where f is a complex-valued function of a complex variable and is defined and suitably restricted in a complete neighborhood of $x=0$ [Koenigs, Ann. Sci. École Norm. Sup (3) **1** (1884), Suppl. 3–41; and Kneser, J. Reine Angew. Math. **187** (1950), 56–57; MR **11**, 726], the present author extends the theory to the two cases where (a) f is defined for $x\geqq 0$ only, and (b) f is

defined and holomorphic in a region $R(\nu, \theta)$, where $x \in R(\nu, \theta)$ if and only if $|x| < \nu$, $|\arg x| < \theta$. The types of restrictions employed are on the order of f as $x \to 0$ in the given set of definition. Let $a = \lim_{x\to 0} x^{-1}f(x)$, for x in the given set of definition; then the three cases $a=0$, $0<a<1$, $a=1$ require separate treatment. In the last case the iterates of f are related to solutions of Abel's functional equation $\lambda(f) = \lambda - 1$, and results of P. Levy [Ann. Mat. Pura Appl. (4) **5** (1928), 269–298] are used. The article includes a fairly extensive survey of previous work on the subject. As a by-product, some results on the asymptotic behavior of sequences $\{f_n\}$ and on asymptotic differentiability of a function defined in a region $R(\nu, \theta)$ are obtained.

W. J. Thron (Boulder, Colo.)

5745:

Wilson, Alan. Sur les familles normales bornées de fonctions holomorphes. C. R. Acad. Sci. Paris **248** (1959), 3391–3392.

Let $\mathscr{F} = \{f(z)\}$ be a family of functions given by $f(z) = \sum a_n(f)z^n$. A necessary and sufficient condition for $f \in \mathscr{F}$ to be holomorphic in $|z| < R$ and $\mathscr{F}$ to be normal there and bounded in the neighborhood of $z=0$, one of these properties not holding for $|z| < R+\varepsilon$ $(\varepsilon > 0)$, is that $\sup |a_n(f)| = A_n < \infty$ for all n, and $\limsup A_n^{1/n} = R^{-1}$.

R is called the radius of regularity of $\mathscr{F}$ and the family is normal-bounded in its circle of regularity. If $R = \infty$, $\mathscr{F}$ is an entire family. For an entire family $\mathscr{F}$ denote $\mathrm{Sup}_{\mathscr{F}} \max_{|z|=r} |f(z)| = \mathscr{M}(r)$. If $\mathscr{M}(r) < \infty$ for all r and $\limsup_{z\to\infty} \log \mathscr{M}(r)/r = H < \infty$ the family is of finite exponential type H [cf. G. R. Johnson, Thesis, Rice Inst., Houston, Texas, 1956]. The function

$$H(\theta): \limsup_{r\to\infty} r^{-1} \log \sup_{\mathscr{F}} |f(re^{i\theta})| = H(\theta), \quad (0 \leq \theta \leq 2\pi),$$

is the indicator function of $\mathscr{F}$. The author considers the three families

$$\mathscr{F} = \{f(z)\} = \left\{\sum \frac{a_n(f)}{n!} z^n\right\}, \qquad \mathscr{G} = \left\{\sum \frac{a_n(f)}{z^{n+1}}\right\},$$
$$\Phi = \{\sum f(n)z^n\}$$

and states the theorem: A necessary and sufficient condition for $\mathscr{F}$ to be of finite exponential type is that $\mathscr{G}$ shall have a finite radius of regularity R, and R is then the type of $\mathscr{F}$.

Two other theorems give conditions for the regularity of Φ, its radius of regularity and its continuation of regularity, as well as necessary and sufficient conditions for the continuation of regularity of a given family.

B. A. Amirà (Jerusalem)

FUNCTIONS OF SEVERAL COMPLEX VARIABLES, COMPLEX MANIFOLDS

See also 5637, 6000.

5746:

Šafeev, M. N. Boundary properties of biharmonic functions. Ukrain. Mat. Ž. **10** (1958), 299–317. (Russian. English summary)

Il s'agit non des fonctions appelées ordinairement biharmoniques mais de fonctions $U(w, z)$ des variables complexes $w = u+iv$, $z = x+iy$ définies dans le bicylindre: $|w| < 1$, $|z| < 1$, deux fois différentiables et satisfaisant à

$$\Delta_1 U = \frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2} = 0, \qquad \Delta_2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

Intégrale double de Poisson n'utilisant pas les frontières $|w| = 1$, $|z| = 1$. Définition convenable des fonctions bisous-harmoniques, qui, lorsqu'elles sont deux fois différentiables, doivent seulement satisfaire à $\Delta_1 U \geq 0$, $\Delta_2 U \geq 0$. Théorèmes de convergence comme celui de Harnack. Représentation des fonctions biharmoniques ≥ 0 par une intégrale double de Poisson-Stieltjes ou Lebesgue. Allure à la frontière et limites radiales p.p. (ou dans certains cônes) s'exprimant comme une certaine dérivée seconde mixte par rapport aux arguments des deux variables complexes.

{À noter qu'on peut réduire les hypothèses de la définition, selon une étude de Avanissian [C. R. Acad. Sci. Paris **244** (1957), 2273–2275; MR **19**, 645].}

M. Brelot (Paris)

5747:

Segre, Beniamino. Proprietà in grande sulle funzioni analitiche di più variabili complesse. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 231–238.

In the space of n complex variables $z_k = x_k + iy_k$, $1 \leq k \leq n$, the author introduces an ad-hoc notion of a semi-regular hypersurface V_{2n-1} and then shows, among other statements, that if for two sets of holomorphic functions $f_m(z)$, $g_m(z)$, $1 \leq m \leq n$, in which the $g_m(z)$ are functionally independent, the linear combination $\sum_{m=1}^n f_m(z)\overline{g_m(z)}$ is 0 on V_{2n-1} (or a little less), then all $f_m(z)$ are $\equiv 0$. This is intended to be a generalization of a theorem of J. J. Kohn [Proc. Amer. Math. Soc. **9** (1958), 175–177; MR **20** #122] that if for holomorphic $f_m(z)$ the linear combination $\sum_{m=1}^n \bar{z}_m f_m(z)$ is 0 on $|z_1|^2 + \cdots + |z_n|^2 = 1$, then all $f_m(z) \equiv 0$. Another generalization of the theorem of J. J. Kohn was given in the meantime by the reviewer [Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 46–47; MR **20** #5297].

S. Bochner (Princeton, N.J.)

5748:

Gottschling, Erhard. Explizite Bestimmung der Randflächen des Fundamentalbereiches der Modulgruppe zweiten Grades. Math. Ann. **138** (1959), 103–124.

Denote by $\mathfrak{H}$ the set of complex, symmetric matrices $Z = X+iY$, of order n, with Y positively definite, and let Γ be the (inhomogeneous) modular group of order n on $\mathfrak{H}$. For every (possibly complex) matrix $W = (w_{kl})$, $k, l = 1, 2, \cdots, n$, denote the absolute value of the determinant of W by abs W. It is known [Siegel, Math. Ann. **116** (1939), 617–657; Amer. J. Math. **65** (1943), 1–86; MR **4**, 242] that one may define a fundamental domain of Γ in $\mathfrak{H}$ by the conditions (i) $|x_{kl}| \leq \frac{1}{2}$; (ii) Y reduced (in the sense of Minkowski); (iii) abs $(CZ+D) \geq 1$. Here (iii) has to be satisfied for all pairs of coprime, symmetric matrices C, D. Siegel has shown [loc. cit.] that the infinite set of inequalities (iii) follows from a finite subset of them, but the exact number of independent inequalities in (iii) was not known for any $n > 1$. For $n = 1$, Γ becomes the ordinary modular group and it is a classical result that a fundamental domain in the plane of the complex variable $z = x+iy$ is defined (essentially) by (i) $|x| \leq \frac{1}{2}$; (ii) $y \geq 0$; (iii) $|z| \geq 1$. In the case $n = 2$,

$$Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix} = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} + i \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix},$$

(i) and (ii) become $|x_i| \leq \frac{1}{2}$ and $y_2 \geq y_1 \geq 2y_3 \geq 0$, respectively. In the present paper the author shows that (iii) contains exactly 19 inequalities, namely $|z_1| \geq 1$, $|z_2| \geq 1$, $|z_1+z_2-2z_3 \pm 1| \geq 1$, and 15 others of the form abs $|Z+S| \geq 1$, where S runs through a set of 15 (explicitly indicated) matrices, whose elements are $+1$, -1, or 0.
E. Grosswald (Princeton, N.J.)

5749:
Seshadri, C. S. Generalized multiplicative meromorphic functions on a complex analytic manifold. J. Indian Math. Soc. (N.S.) **21** (1957), 149–178.

Soient X une variété analytique complexe, k_P le corps des germes de fonctions méromorphes au point $P \in X$; on appelle germe de diviseur (à gauche) de type m en P, une classe d'équivalence à gauche de $GL(m, k_P)$ modulo les matrices inversibles de germes de fonctions holomorphes en P. On en déduit la notion de diviseur global. On appelle fonction méromorphe multiplicative (en abrégé f.m.m.) généralisée associée à une représentation χ du groupe fondamental $\pi_1(X)$ de X dans $GL(m, C)$ une matrice $F(z)$ de fonctions méromorphes sur le revêtement universel $\tilde{X}$ de X, à déterminant non identiquement nul et telle que: $F(Sz) = \chi(S)F(z)$, $S \in \pi_1(X)$, $z \in \tilde{X}$. Théorème: Tout diviseur sur une surface de Riemann non compacte est le diviseur d'une f.m.m. généralisée. Il en résulte: dans le plan complexe C ou le disque unité D, tout diviseur est le diviseur d'une matrice de fonctions méromorphes, donc: tout fibré vectoriel sur C ou sur D est analytiquement trivial. En utilisant un lemme de G. D. Birkhoff [Math. Ann. **74** (1913), 122–133], l'auteur déduit de cela que tout fibré vectoriel sur la sphère de Riemann est la somme directe de fibrés vectoriels à fibre de dimension un, d'où la classification des fibrés vectoriels sur la sphère de Riemann déjà donnée par A. Grothendieck [Amer. J. Math. **65** (1957), 121–138; MR **19**, 315]. La technique de démonstration est reliée à celle de A. Weil [J. Math. Pures Appl. **17** (1938), 47–87]; les résultats suivants, établis par l'auteur, sont aussi utilisés: tout fibré vectoriel sur une variété de Stein ou une variété algébrique projective est défini par un diviseur. L'auteur caractérise, d'autre part, les diviseurs (de type un) de f.m.m. Une généralisation de la notion de fonction multiplicative C^∞ sur une variété différentiable X conduit à l'étude des formes différentielles à coefficients dans un faisceau localement simple. Pour ces formes, l'auteur établit un "théorème de de Rham", un théorème de dualité et, dans le cas où X est riemannienne compacte, pour toute forme ω du type ci-dessus, il montre qu'il existe une décomposition unique $\omega = H\omega + \omega_1 + \omega_2$ où $H\omega$ est harmonique, $\omega_1 = d\omega_1'$ et $\omega_2 = \partial\omega_2'$.
P. Dolbeault (Bordeaux)

SPECIAL FUNCTIONS

See also 5540, 5721, 5876, 6077.

5750:
Carlitz, L. Note on the integral of the product of several Bernoulli polynomials. J. London Math. Soc. **34** (1959), 361–363.

The product $B_m(x)B_n(x)$ of two Bernoulli polynomials may be expressed as a linear combination of Bernoulli polynomials of the form $B_{m+n-2r}(x)$. Multiplying this relation by $B_p(x)$ and using the known formula

$$\int_0^1 B_p(x)B_q(x)dx = (-1)^{p+1}\frac{p!q!}{(p+q)!}B_{p+q},$$

the author evaluates the integral $\int_0^1 B_m(x)B_n(x)B_p(x)dx$ in terms of Bernoulli numbers. A formula for the product of four Bernoulli polynomials is also given and another method is indicated for evaluating the integral of the product of any finite number of Bernoulli polynomials.
T. M. Apostol (Pasadena, Calif.)

5751:
Gandhi, J. M. Some integrals for Genocchi numbers. Math. Mag. **33** (1959/60), 21–23.

Put

$$\frac{2x}{e^x+1} = \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}.$$

The author obtains the following two results:

$$\int_0^\infty x^{2n-2} \log \coth \frac{\pi x}{2}\, dx = (-1)^n \frac{G_{2n}\pi}{4n(2n-1)},$$

$$\int_0^\infty x^{2n} \coth \pi x \operatorname{cosech} \pi x\, dx = (-1)^n \frac{G_{2n}}{2\pi}.$$

L. Carlitz (Durham, N.C.)

5752a:
Robin, Louis. Extension aux fonctions de Legendre de première espèce de résultats relatifs aux fonctions coniques de Mehler. C. R. Acad. Sci. Paris **249** (1959), 1081–1083.

5752b:
Robin, Louis. Nouvelles extensions aux fonctions de Legendre de première espèce de résultats relatifs aux fonctions coniques de Mehler. C. R. Acad. Sci. Paris **249** (1959), 1178–1179.

Some formulae involving integrals of the Mehler conical functions [A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions I*, McGraw-Hill, New York-Toronto-London, 1953; MR **15**, 419; p. 174, formulae (6), (7), (8), (9)]; are extended to the Legendre functions $P_n(\nu)$ $(-1 < \mathrm{Re}\, n < 0)$ by a change of variable.
C. A. Swanson (Vancouver, B.C.)

5753:
★Slater, L. J. Confluent hypergeometric functions. Cambridge University Press, New York, 1960. xi+247 pp. $12.50.

About one half of this volume contains an account of those properties of confluent hypergeometric functions of interest to applied mathematicians and the other half, numerical tables.

The contents of the theoretical part is as follows. Chapter 1. The differential equations, their solutions and relations between these. Chapter 2. Differentiation formulas, recurrence relations, and other functional relations (including some expansions). Chapter 3. Integral representations and integral formulas. Chapter 4. Asymptotic

expansions. Chapter 5. Related differential equations and particular cases of confluent hypergeometric functions. Chapter 6. Descriptive properties, zeros, computation of confluent hypergeometric functions.

The scope of the work is thus similar to that of Tricomi's well-known book [*Funzioni ipergeometriche confluenti*, Edizioni Cremonese, Rome, 1954; MR **17**, 967]. There are some considerable differences, though. One of these is a very useful feature of this book not found elsewhere: throughout the work Kummer's functions and Whittaker's functions are considered side by side. Another is a very full discussion of asymptotic representations and expansions, including some new results of the author. {Although this discussion occupies about one third of the text, it is not a complete account of the subject; the work of N. Kazarinoff [Trans. Amer. Math. Soc. **78** (1955), 305–328; J. Math. Mech. **6** (1957), 341–360; MR **16**, 695; **19**, 133], Erdélyi and Swanson [Mem. Amer. Math. Soc. no. 25 (1957); MR **19**, 850] and some other works are not even mentioned.} A third useful feature is the inclusion of some material on generalized hypergeometric series.

The second half of the volume is devoted to numerical tables. The first table gives 7D values of the smallest positive zero of ${}_1F_1(a;b;x)$ for $a=-4(.1)-.1$ and $b=.1(.1)2.5$; the second, 7 or 8S values of ${}_1F_1(a;b;x)$ for $a=-1(.1)1$, $b=.1(.1)10$, and $x=.1(.1)10$; and the third, 7D values of ${}_1F_1(a;b;1)$ for $a=-11(.2)2$ and $b=-4(.2)1$.

A. Erdélyi (Pasadena, Calif.)

5754:

Luke, Yudell L. Expansion of the confluent hypergeometric function in series of Bessel functions. Math. Tables Aids Comput. **13** (1959), 261–271.

The author expands the confluent hypergeometric function in a series of (modified) Bessel functions of integral order. He also discusses other expansions of this kind (due to Buchholz and Tricomi) and considers the special forms of these expansions for incomplete gamma functions, the error function, exponential integral, and related functions. He concludes with examples illustrating the application of these expansions to the numerical computation of confluent hypergeometric functions and their special cases.

A. Erdélyi (Pasadena, Calif.)

5755:

Chako, N.; and Meixner, J. A characterization of hypergeometric functions. Arch. Rational Mech. Anal. **4**, 89–96 (1959).

Truesdell [*An essay toward a unified theory of special functions*, Princeton Univ. Press, 1948; MR **9**, 431] has shown that a large number of special functions satisfy a functional equation of the form $\partial F(x,\alpha)/\partial x=F(x,\alpha+1)$, which he called the F-equation. The authors show that, among this large number, hypergeometric functions can be characterized as follows: they satisfy two functional equations reducible to the F-equation, one for ascending and the other for descending α, and the system of two F-equations which they satisfy has two linearly independent solutions.

A. Erdélyi (Pasadena, Calif.)

5756:

Sharma, K. C. Infinite integrals involving E-functions. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 161–165.

The integrals

$$\int_0^\infty (t^2+2pt)^{-\mu/2}(p+t)^{-k}E[l\,;\alpha_r:m\,;\beta_s:z(p+t)]P_\nu^\mu \times(1+t/p)dt,$$

$$\int_0^\infty x^{\lambda-1}K_\nu(px)E[l\,;\alpha_r:m\,;\beta_s:z/\sqrt{x}]dx$$

are evaluated as finite sums of E-functions multiplied by powers of p and z. The derivations employ operational calculus and various integral representations of similar sums of E-functions.

N. D. Kazarinoff (Madison, Wis.)

5757:

MacRobert, T. M. Infinite series of E-functions. Math. Z. **71** (1959), 143–145.

"A number of series of E-functions are summed by expressing the E-functions as Barnes integrals and interchanging the order of integration and summation." (From the introduction)

N. D. Kazarinoff (Madison, Wis.)

5758:

MacRobert, T. M. Integrals of products of E-functions. Math. Ann. **137** (1959), 412–416.

Let l, m, n, p and q be natural numbers such that $l\geqq n+1$ and $p\geqq q+1$. The integrals evaluated are

$$\int_0^\infty \lambda^{k-1}E(l\,;\beta_u:n\,;\sigma_v:z/\lambda)E(p\,;\alpha_r:q\,;\rho_s:\lambda^m)d\lambda$$

$$(\Re(m\alpha_r+k)>0,\quad r=1,2,\cdots,p;\quad \Re(\beta_u-k)>0,\quad u=1,2,\cdots,l;\quad |\arg z|<\tfrac{1}{2}(l-n+1)\pi)$$

and

$$\int_0^\infty \lambda^{-k-1}E(l\,;\beta_u:n\,;\sigma_v:\lambda z)E(p\,;\alpha_r:q;\rho_s:\lambda^m)d\lambda$$

$$(\Re(k)>0;\quad \Re(m\alpha_r+\beta_u-k)>0,\quad r=1,2,\cdots,p,\quad u=1,2,\cdots,l;\quad |\arg z|<\tfrac{1}{2}(l-n+1)\pi).$$

The evaluations are in terms of finite sums of products of trigonometric functions and E-functions, the latter involving a "large number of parameters of different types".

N. D. Kazarinoff (Madison, Wis.)

5759:

Kreyszig, Erwin; and Todd, John. On the radius of univalence of the function $\exp z^2 \int_0^z \exp(-t^2)dt$. Pacific J. Math. **9** (1959), 123–127.

It is shown that the analytic function $f(z)$ quoted in the title is univalent in the circle $|z|<\rho=0.924\cdots$, where ρ is the positive solution of $2\rho\exp(-\rho^2)\int_0^\rho \exp(t^2)dt=1$. Since $f'(i\rho)=0$, this value is the best possible.

Z. Nehari (Pittsburgh, Pa.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 5876, 6194.

5760:

★Boole, George. A treatise on differential equations. 5th ed. Chelsea Publishing Company, New York, 1959. xxiv+735 pp.

Consists of the 4th ed. [Macmillan, London, 1877] with the Supplementary Volume [University Press, Cambridge, 1865] appended and paged consecutively to it.

5761:

Koŝlyakov, N. S.; and Guseva, N. K. A third order ordinary differential equation of Laplace type. Inž.-Fiz. Ž. 1 (1958), no. 5, 71–75. (Russian)

This paper deals with the explicit solution of $xy''' + xy' + \nu y = 0$ (and thus of $x^3W''' - 3\nu x^2W'' + [3\nu(\nu+1)+x^2]xW' - \nu(\nu+1)(\nu+2)W = 0$) by the use of Laplace integrals.

R. R. D. Kemp (Kingston, Ont.)

5762:

Koŝlyakov, N. S.; and Maksimova, I. G. A fourth order ordinary differential equation of Laplace type. Inž-Fiz. Ž. 1 (1958), no. 8, 73–83. (Russian)

This paper uses the method of Laplace integrals to discuss (and calculate) explicit solutions of a special case of the fourth order linear equation with linear coefficients. The authors seem to be solving $xy^{\mathrm{IV}} + 3y''' - y'' + y' - (x+1)y = 0$, rather than the equation they give.

R. R. D. Kemp (Kingston, Ont.)

5763:

Turrittin, H. L. Linear differential or difference equations with constant coefficients. Amer. Math. Monthly **66** (1959), 869–875.

Let (1) $dX/dt = AX$ [(2) $X(t+h) = AX(t)$] be a system of linear differential [difference] equations, A a real constant $n+n$ matrix. To find the general solution of (1) or (2) without excursion into the complex field, (1) and (2) are transformed by reducing A to a Jordan-like canonical form $(B_1, \cdots, B_p, G_1, \cdots, G_q)$, where B_i, corresponding to real eigenvalues, are standard Jordan blocks, and each G_j, corresponding to a pair of conjugate complex eigenvalue pairs $\mu + \nu i$ has 2×2 blocks $\begin{pmatrix}\mu & -\nu\\ \nu & \mu\end{pmatrix}$ in the diagonal positions, and 2×2 identity matrices in the subdiagonal positions. The general solutions of the transformed versions of (1) and (2) are explicitly given in terms of solution blocks corresponding to the B_i and G_j. {An apparent misprint erroneously gives the general solutions of (1) and (2) as $X = PY_f(t) + K$ rather than $X = PY_f(t)K$, Y_f being a matrix of fundamental solutions.}

R. F. Rinehart (Durham, N.C.)

5764:

Kac, I. S. Density of the spectrum of a string. Dokl. Akad. Nauk SSSR **126** (1959), 1180–1182. (Russian)

The author considers a string stretched along the x-axis from $x=0$ to $x=L$ and defines $M(x)$ to be the mass of the string on the interval $[0, x)$. The spectrum of the string is then defined to be the squares of the frequencies of free vibration, and if these are $(0 \leqq)\ \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, conditions are given for the convergence of $\sum_{j=1}^{\infty} \lambda_j^{-\alpha}$ for various values of α.

R. R. D. Kemp (Kingston, Ont.)

5765:

Datzeff, A. B. Über die Eigenwerte einiger Differentialgleichungen. I. C. R. Acad. Bulgare Sci. **12** (1959), 113–116. (Russian summary)

Es handelt sich um die Bestimmung der Eigenwerte und Eigenfunktionen der linearen Differentialgleichung zweiter Ordnung $y'' + f(x)y = 0$ mit den Randbedingungen $y(0) = 0$, $y(l) = 0$. Die Funktion $f(x)$ hängt von einem Parameter μ ab, $f(x) = \Phi(x) + \mu\varphi(x)$, $0 \leqq x \leqq l$. Die Funktionen $\Phi(x)$ und $\varphi(x)$ sind in ihrem Definitionsbereich positivwertig und sind beschränkte integrable Funktionen. Zunächst wird das Cauchysche Problem für die Anfangsbedingungen $y(x_0) = y_0$, $y'(x_0) = y_0'$, $x > x_0$ gelöst. Dabei verwendet der Verfasser ein lineares Approximationsverfahren, das er bereits in seiner Dissertation benutzt hat [vgl. A. Datzeff, Ann. Physique **10** (1938), 583–673]. Das Ergebnis läßt sich für die Lösung der ursprünglich gestellten Aufgabe verwenden und führt auf eine schematische Darstellung der gesuchten Eigenwerte. Mit der gleichen Methode lassen sich auch Fälle allgemeinerer Randbedingungen behandeln.

M. Pinl (Cologne)

5766:

Everitt, W. N. Integrable-square solutions of ordinary differential equations. Quart. J. Math. Oxford Ser. (2) **10** (1959), 145–155.

Let L be an ordinary differential operator of order n $(n \geqq 1)$

$$L = a_0D^n + a_1D^{n-1} + \cdots + a_n,$$

where $a_0, \cdots, a_n$ are complex-valued continuous functions on $0 \leqq x < \infty$ with $a_0(x) \neq 0$ there. It is assumed that L is formally self-adjoint. For any complex number λ let $E(\lambda)$ denote the eigenspace consisting of all $\varphi \in L^2(0, \infty)$ satisfying $L\varphi = \lambda\varphi$, and let $S(\lambda) = \dim E(\lambda)$. It is shown that if $n = 2\nu$ $(\nu \geqq 1)$ then $S(\lambda) \geqq \nu$ for all λ such that $\operatorname{Im}\lambda \neq 0$, and if $n = 2\nu - 1$ $(\nu \geqq 1)$ then either (i), for all λ with $\operatorname{Im}\lambda \neq 0$, $S(\lambda) \geqq \nu - 1$ $(\operatorname{Im}\lambda > 0)$, $S(\lambda) \geqq \nu$ $(\operatorname{Im}\lambda < 0)$, or (ii), for all λ with $\operatorname{Im}\lambda \neq 0$, $S(\lambda) \geqq \nu$ $(\operatorname{Im}\lambda > 0)$, $S(\lambda) \geqq \nu - 1$ $(\operatorname{Im}\lambda < 0)$. The inequalities are best possible. In case n is odd the alternative (i) is valid when $(-i)^n a_0(x) > 0$ $(0 \leqq x < \infty)$, and (ii) is valid when $(-i)^n a_0(x) < 0$ $(0 \leqq x < \infty)$. The proof makes use of a detailed study of appropriate Gram matrices.

E. A. Coddington (Los Angeles, Calif.)

5767:

Swanson, C. A. Asymptotic perturbation series for characteristic value problems. Pacific J. Math. **9** (1959), 591–608.

This paper represents an extension, to differential operators of order $n \geqq 2$, of some results obtained previously by the author for the case $n = 2$ [J. Math. Mech. **6** (1957), 823–846; MR **19**, 1054]. Let $L = p_nD^n + p_{n-1}D^{n-1} + \cdots + p_0$ be an ordinary differential operator on the interval $0 < x \leqq b$ $(b > 0)$, with the p_j real-valued, of class C^j, and $p_n(x) \neq 0$ there. The basic operator in $L^2(0, b)$ is L defined for suitable functions satisfying boundary conditions. A perturbed operator A_ε is obtained for $0 < \varepsilon < b$ by considering L on $L^2(\varepsilon, b)$ with domain consisting of suitable functions satisfying homogeneous boundary conditions at $x = \varepsilon$ and $x = b$. It is shown that for each isolated eigenvalue Λ of the basic operator there is an eigenvalue $\lambda(\varepsilon)$ of A_ε which tends to Λ as $\varepsilon \to 0$, and moreover λ can be represented by an asymptotic expansion as $\varepsilon \to 0$. An asymptotic expansion for an eigenfunction u corresponding to λ is also established of the form $u(x, \varepsilon) = U(x)[1 + o(1)]$. Here U is an eigenfunction for Λ,

and the asymptotic form is valid uniformly for x in a certain closed subset of $\varepsilon \leq x \leq b$ as $\varepsilon \to 0$.
E. A. Coddington (Los Angeles, Calif.)

5768:

Berkowitz, Jerome. On the discreteness of spectra of singular Sturm-Liouville problems. Comm. Pure Appl. Math. **12** (1959), 523–542.

Consider the differential equation $Lu \equiv r^{-1}(-(pu')'+qu)=\lambda u$ on $-\infty \leq x_- < x < x_+ \leq \infty$ where L has domain D contained in $L^2(x_-, x_+)$. Let D_0 be a domain of functions with piecewise continuous second derivatives and identically zero near the end-points. Let L be symmetric on D_0 or else denote some self-adjoint extension on D and define accordingly, for $n=1, 2, \cdots$, κ_n or λ_n to be $\min_{S_n} \max_{u \in S_n} (u, Lu)$ with $(u, u)=1$ and, where in the definitions of κ_n or λ_n, respectively, S_n denotes an n-dimensional subspace of D_0 or of D. It is shown, among other things, that if L is self-adjoint on D, then L is bounded from below if and only if $\lambda_n > -\infty$, in which case $\lambda_1 \leq \lambda_2 \leq \cdots \to \lambda_* \ (\leq \infty)$ and if $\lambda_n < \lambda_*$, then λ_n is the nth eigenvalue of L. In the general case, assuming $p', q, p > 0$, $r > 0$ to be real and piecewise continuous on the compact subintervals of the fundamental interval, it is shown that $\lambda_n \leq \kappa_n \leq \lambda_{n+2}$. As a consequence there is obtained Rellich's result that if L on D_0 is bounded from below, then so are its self-adjoint extensions on D. Results dealing with connections between the discreteness of the spectra and the number of zeros of solutions of $Lu+\lambda u$ on the fundamental interval are obtained.
C. R. Putnam (Lafayette, Ind.)

5769:

Halanaĭ, A.; and Šandor, Št. Sturm-type theorems for self-conjugate systems of linear differential equations of higher order. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1 (49)** (1957), 401–431. (Russian)

In this paper the authors are concerned with oscillation and comparison theorems for self-adjoint systems of nth order equations. The basic system under consideration is (1) $\sum_{i=0}^{n} (-1)^{n-i} D^i(\theta_{n-i} D^i y)=0$, where $\theta_i(t)$ is a real symmetric matrix of order p and y is a p-dimensional vector. It is assumed throughout that $\theta_0 > 0$ and that $\theta_i \in C^{n-i}$. This system is transformed in a special manner to a canonical type of first order system, and most of the theorems are proved in terms of such systems. Here we shall restate some of the main theorems in terms of (1).

Two points are said to be conjugate for (1) if there is a non-zero solution of (1) which vanishes together with its first $n-1$ derivatives at both of the points. The separation theorem states that two systems of conjugate points either coincide or separate each other. The comparison theorem compares the systems of points $\{t_k\}$ and $\{\tau_k\}$ conjugate to $t_0=\tau_0$, which arise from two systems of the form (1) with coefficients θ_i and φ_i respectively. If $\theta_i \leq \varphi_i$ $(i=0, 1, \cdots, n)$, then $t_k \leq \tau_k$ for $k > 0$ and $t_k \geq \tau_k$ for $k < 0$. Note that the t_k's and τ_k's are arranged in order: $t_k < t_{k+1}$, $\tau_k < \tau_{k+1}$.

The authors then go on to consider systems depending on a parameter. This essentially amounts to considering an eigenvalue problem for $Ly=My$ where L is the operator in (1) and M is a similar operator with the highest order term missing. The boundary conditions are $D^i y(a)=D^i y(b)=0$ for $i=0, 1, \cdots, n-1$; and it is assumed that all the coefficient matrices in M are non-negative with at least one strictly positive. Under these conditions it is proved that there is an increasing sequence of eigenvalues λ_k with $\lim_k \lambda_k = \infty$. Both algebraic and variational methods are used in the proofs.
R. R. D. Kemp (Kingston, Ont.)

5770:

Singh, S. N. The determination of eigen-functions of a certain Sturm-Liouville equation and its application to problems of heat-transfer. Appl. Sci. Res. A **7** (1958), 237–250.

The subject is the generalized Sturm-Liouville system

$$\frac{d}{dx}\{p(x)y'\}+\{q(x)+r(x, \lambda)\}y = 0,$$

subject to the boundary conditions $y'(a)=h_1 y(a)$, $y'(b)=h_2 y(b)$. The solution is made to depend on the auxiliary equation

$$\frac{d}{dx}\{p(x)y'\}+g(x, \alpha)y = 0,$$

where α is a parameter. The form of $g(x, \alpha)$ depends on $p(x)$ in a manner which is not made clear, but in the examples $g(x, \alpha)=\alpha^2$. By expressing the problem in terms of solutions of the auxiliary equation it is reduced to the solution of an infinite set of simultaneous algebraic equations. The method is applied to three problems of heat transfer.
E. C. Titchmarsh (Oxford)

5771:

Constantinescu, Florin. Théorèmes du type Sturm pour les extrêmes des intégrales d'une équation différentielle linéaire et homogène du deuxième ordre. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 267–274. (Romanian. Russian and French summaries)

L'A. prendendo in considerazione il wronskiano di due soluzioni indipendenti $y_1(x)$, $y_2(x)$ dell'equazione $y''+f(x)y'+g(x)y=0$ con $f(x)$, $g(x)$ continue in $[a, b]$, $g(x)>0$, e l'identità di Picone tra due soluzioni $y(x)$, $z(x)$ rispettivamente delle equazioni $[f_1 y']'+g_1 y=0$, $[f_2 z']'+g_2 z=0$, con f_1, f_2, g_1, g_2 continue in $[a, b]$, $f_1 \geq f_2 > 0$, $g_2 \geq g_1 > 0$, dimostra nel primo caso un teorema di separazione degli zeri di y_1', y_2' e nel secondo caso un teorema di esistenza di uno zero di z' fra due zeri di y'.
G. Sansone (Firenze)

5772:

Langer, Rudolph E. Asymptotic theories for linear ordinary differential equations depending upon a parameter. J. Soc. Indust. Appl. Math. **7** (1959), 298–305.

A brief recapitulation of the theory of asymptotic solution of ordinary differential equations whose modern development was initiated by the author is given with reference to unsolved turning-point problems for nth order equations.
N. D. Kazarinoff (Madison, Wis.)

5773:

Turrittin, H. L.; and Harris, W. A., Jr. Standardization and simplification of systems of linear differential equations involving a turning point. J. Soc. Indust. Appl. Math. **7** (1959), 316–324.

The systems studied are of the form

$$\varepsilon^h \frac{dX}{dt} = A(t, \varepsilon)X$$

where h is a positive integer, ε is a small parameter, and X and A are $n\times n$ matrices with $A=\sum_0^\infty \varepsilon^k A_k(t)$ ($|\varepsilon|<\varepsilon_0$, $|t|<t_0$). The progress thus far made on turning problems for such systems is reviewed. The main results are matrix transformation methods which reduce the general problem to a large number of canonical problems.

N. D. Kazarinoff (Madison, Wis.)

5774:

Yakubovič, V. A. The small parameter method for canonical systems with periodic coefficients. J. Appl. Math. Mech. **23** (1959), 17–43 (15–34 Prikl. Mat. Meh.).

Consider the system

$$(1) \qquad x' = [C+\varepsilon B(\theta t, \varepsilon)]x, \quad t\geqq 0,$$

where C is a constant matrix, $\theta>0$ is a constant, ε is a small parameter, $B(\tau+2\pi, \varepsilon)=B(\tau, \varepsilon)$ is analytic in ε and there exists a function $\eta(\tau)$, L-integrable in $[0, 2\pi]$, such that $\|B(\tau, \varepsilon)\|\leqq \eta(\tau)$, $0\leqq\tau\leqq 2\pi$, $0\leqq|\varepsilon|\leqq\varepsilon_0$. A method of successive approximations is given to determine the characteristic exponents of (1) in the case where some of the eigenvalues of C are congruent (mod $i\theta$). Since the exponents are only determined mod $i\theta$, a transformation is made to reduce (1) to

$$(2) \qquad dy/d\tau = [K_0+\varepsilon D(\tau, \varepsilon)]y, \quad \tau\geqq 0,$$

where no distinct eigenvalues of K_0 are congruent (mod i). The exponents of (2) for $\varepsilon=0$ may now be chosen as the eigenvalues of K_0. Matrices $K(\varepsilon)$, $P(\tau, \varepsilon)=P(\tau+2\pi, \varepsilon)$, $P(0, \varepsilon)=I$, are determined so that

$$Y(\tau, \varepsilon) = P(\tau, \varepsilon)\exp[K(\varepsilon)\tau]$$

is a fundamental system of absolutely continuous solutions of (2). These results are applied to the case where (1) is canonical. This case is difficult since the solution of (1) will be bounded only if the eigenvalues of $K(\varepsilon)$ are purely imaginary. Therefore, in general, the complete expansion of $K(\varepsilon)$ will need to be known. However, the author proves the following result which is sufficient for most practical cases. Theorem: Let $K(\varepsilon)=K_0+\varepsilon K_1+\varepsilon^2K_2+\cdots$ and let $K^{(m)}(\varepsilon)=K_0+\varepsilon K_1+\cdots+\varepsilon^m K_m$. If there exists an m such that the eigenvalues of $K^{(m)}(\varepsilon)$ are purely imaginary and distinct then the eigenvalues of $K(\varepsilon)$ are purely imaginary and distinct for ε sufficiently small. This theorem is illustrated by an example of fourth order. There are some misprints in the paper, the most serious being the omission of D_n at the end of formula (3.2). [Related papers: Cesari, Atti. Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11** (1941), 633–695; MR **8**, 208; Hale, Riv. Mat. Univ. Parma **5** (1954), 137–167; MR **17**, 36; Gambill, ibid. **6** (1955), 37–43; MR **17**, 849; Hale, Illinois J. Math. **2** (1958), 586–591; MR **21** #1417; Golomb, Arch. Rational Mech. Anal. **2** (1958), 284–308; MR **21** #3624; Yakubovič, Vestnik Leningrad. Univ. **13** (1958), no. 13, 35–63; MR **21** #749.]

J. K. Hale (Baltimore, Md.)

5775:

Utz, W. R. Boundedness of solutions of a linear equation. Monatsh. Math. **63** (1959), 356–358.

Let $x''+(a/t)x'+q(t)x=0$ where $a\geqq 0$ and $q(t)$ is differentiable for $t\geqq t_0>0$. If $q'(t)\leqq 0$ and $t^a q(t)>d>0$ for all $t\geqq t_0$, and if $a\leqq 2$, then $x(t)=O(1)$ as $t\to\infty$; if $q'(t)\geqq 0$ for all $t\geqq t_0$ and $q(t)>0$ for some $t\geqq t_0$, then $x(t)=o(1)$ as $t\to\infty$.

H. A. Antosiewicz (Los Angeles, Calif.)

5776:

Suyama, Yukio. On the oscillation of the second ordered linear differential equation. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 30–36.

The author presents a general oscillation criterion, which for the differential equation

$$(*) \qquad y''+q(x)y = 0$$

leads to the following results: (1) if $q(x)$ is positive and continuous on $T\leqq x<\infty$, then (*) is oscillatory whenever $\int_x^\infty q(t)dt<\infty$ and $\liminf_{x\to\infty} q(x)[\int_x^\infty q(t)dt]^{-2}>4$; (2) if $q(x)$ is continuous and real-valued with $0<\int_x^\infty q(t)dt<\infty$ for $T\leqq x<\infty$, then (*) is oscillatory whenever

$$\limsup_{x\to\infty} q(x)\left[\int_x^\infty q(t)dt\right]^{-2} < 4.$$

In conclusion there is given a sufficient condition for (*) to be non-oscillatory which involves the assumption that $\int_x^\infty dt\int_t^\infty|q(s)|ds<\infty$ for $T\leqq x<\infty$. It is to be remarked that in case $q(x)\geqq 0$ on $T\leqq x<\infty$ an elementary argument shows that the above conclusions (1) and (2) are consequences of the respective results of corollary 1 to theorem 7, and theorem 5 of Hille, Trans. Amer. Math. Soc. **64** (1948), 234–252 [MR **10**, 376].

W. T. Reid (Iowa City, Iowa)

5777:

Reid, William T. Generalized linear differential systems. J. Math. Mech. **8** (1959), 705–726.

The author generalizes the linear second order system

$$(1) \qquad (R(x)u')'-P(x)u = 0$$

to a system

$$(2) \qquad R(x)u' = v, \qquad dv = (dM(x))u,$$

where the last part of (2) is to be interpreted as $v(x)=\int_a^x(dM(t))u(t)+v(a)$ in the sense of Riemann-Stieltjes. The systems (2) contain the scalar equations of the type treated by Feller [Illinois J. Math. **1** (1957), 198–286; MR **19**, 1052]. In (2), it is supposed that $R(x)$ is a non-singular matrix such that $R(x)$, $R^{-1}(x)$ are Lebesgue summable, $M(x)$ is of bounded variation, and a solution vector $u(x)$ is required to be absolutely continuous (so that $v(x)$ is of bounded variation). By successive approximations, an existence and uniqueness theorem for initial value problems is obtained. For the self-adjoint case, where $R=R^*$, $M=M^*$ are Hermitian and R positive definite, the author extends results known for (1) to (2). These involve the connection between the positive definiteness of the quadratic form

$$I(\eta) = \int_a^b \eta'^*R\eta'dx + \int_a^b \eta^*(dM)\eta$$

and the property of (2) being disconjugate on $[a, b]$; criteria for (2) to be non-oscillatory on $(a, b]$; the author's notion of a principal solution of (1); and generalized Sturm oscillation and comparison theorems (of the type derived by Morse for real systems (1)). The last part of the paper deals with a system of the form

$$(3) \qquad \begin{aligned} R(x)u'+Q_1(x)u &= v,\\ dv &= (Q_2(x)u'+P(x)u)dx+(dM(x))v\end{aligned}$$

and a reduction of the theory of (3) to that of (2).

P. Hartman (Baltimore, Md.)

5778:

Corduneanu, C. Sopra i problemi ai limiti per alcuni sistemi di equazioni differenziali non lineari. Rend. Accad. Sci. Fis. Mat. Napoli (4) **25** (1958), 98–106.

L'A. studia alcuni problemi ai limiti per sistemi di equazioni differenziali ordinarie non-lineari. Fra i vari risultati riportiamo il seguente: l'equazione: $y''=f(x, y)$, con $f(x, y)$ continua insieme alla derivata prima rispetto ad y per $a \leqq x \leqq b$, $-\infty < y < +\infty$ e tale che: $\partial f/\partial y \geqq m > 0$, ammette una ed una sola soluzione in (a, b) per cui:

$$(1) \qquad \alpha' y'(a)+\alpha y(a) = A; \qquad \beta' y'(b)+\beta y(b) = B$$

con: $\alpha'^2+\alpha^2>0$, $\alpha'\alpha \leqq 0$, $\beta'^2+\beta^2>0$, $\beta'\beta \geqq 0$. Sono anche considerati altri problemi ove le (1) sono sostituite da condizioni non-lineari. *G. Stampacchia* (Genoa)

5779:

Šaršanov, A. A. Extension of Floquet's theorem to non-linear equations. Dokl. Akad. Nauk SSSR **127** (1959), 1179–1182. (Russian)

For the first order linear systems Floquet's theorem answers in the affirmative the following question. Can a system that depends periodically on time be transformed into an autonomous system such that the values of a solution of the latter coincide at integral multiples of the period with those of a solution of the former, provided their initial values coincide? The author shows that for certain 2-dimensional non-linear first-order systems the answer is also in the affirmative.

H. A. Antosiewicz (Los Angeles, Calif.)

5780:

Jones, John, Jr. On monotone and positive solutions of second order nonlinear differential equations. Proc. Amer. Math. Soc. **10** (1959), 570–573.

Two theorems are proved concerning the equation

$$y''+f(y, y')y'+g(y) = 0, \quad y' = dy/dx.$$

(1) If $f(y, y') \geqq 0$ for all real y, y', if $yg(y) \geqq 0$, if $\int_0^y g(z)dz \to \infty$ as $y \to \infty$, and if

$$\int_E [f(y, y')+g(y)/y']dx \neq \infty$$

where $E=\{x \mid y(x) \geqq 0,\ y'(x)>0\}$, then a solution $y(x)$, valid for all large x, approaches zero monotonically as x tends to infinity. (2) If $f(y, y') \geqq 0$ for all real y, y', if $yg(y) \geqq 0$, if $|f(y, y')y'| \leqq g(y)$ for all real y, y', and if there exist numbers a and b such that $a<b$ and

$$\int_b^\infty (x-a)[f(y, y')y'+g(y)]dx = \infty,$$

$[f(y, y')y'+g(y)]^{1/2} \geqq [y(x)]^2$, then each solution of the equation not identically zero is oscillatory.

W. R. Utz (Columbia, Mo.)

5781:

Sansone, Giovanni. Sopra un'equazione che si presenta nella determinazione delle orbite in un sincrotrone. Rend. Accad. Naz. dei XL (4) **8/9** (1957/58), 99–172.

The equation considered is of the form (*) $\ddot{x}+\varphi(x)x=p(t)$ where originally $p(t)=\sum_{r=1}^{b}(a_r \sin rt+b_r \cos rt)$ and $\varphi(x)=\lambda^2+\varphi_1(x)$ with $\varphi_1(x)\equiv 0$ for $-a \leqq x \leqq a$ $(a>0)$, $\varphi_1(x)=m(x-a)$ for $x \geqq a$, $\varphi_1(x)=m(x+a)$ for $x=-a$. However all results are proved under more general assumptions about $p(t)$ and $\varphi(x)$. Let $x(t)$ be the function constructed in the following way: for $0 \leqq t < \pi/2$, $x(t)$ satisfies (*) and the initial conditions $x(0)=x_0$, $\dot{x}(0)=y_0$, and for $k\pi/2 \leqq t \leqq (k+1)\pi/2$ $(k=1, 2, \cdots)$ $x(t)$ satisfies again (*) and the initial conditions

$$x(k\pi/2) = \lim_{t\to\frac{1}{2}\pi-0} x((k-1)\pi/2+t)+c\cdot \lim_{t\to\frac{1}{2}\pi-0} \dot{x}((k-1)\pi/2+t),$$

$$\dot{x}(k\pi/2) = \lim_{t\to\frac{1}{2}\pi-0} \dot{x}((k-1)\pi/2+t),$$

where c is a given constant. The main problem studied in the paper consists in looking for points (x_0, y_0) such that the function $x(t)$ is defined for $t \in (0, \infty)$ and remains there bounded. These points are called to have the property (P). In § 1 (*) has a more general form $\ddot{x}+f(t)\varphi(x)x=p(t)$ and under certain assumptions the author determines a positive constant A_1 such that the point (x_0, y_0) with $x_0 \leqq -A_1$, $y_0 \leqq 0$ has not the property (P), more exactly that there exists a finite time t_1 such that $\lim_{t\to t_1-0} x(t)=-\infty$, $\lim_{t\to t_1-0} \dot{x}(t)=-\infty$. In § 2 there are studied the characteristics of the equation (*) with $p(t)=\text{const}$ in the phase plane. § 3 contains many results concerning the main problem. All of them have the same character: there are determined regions whose points have not the property (P), more exactly, the respective function $x(t)$ has the property that $\lim_{t\to T-0} x(t)=-\infty$, $\lim_{t\to T-0} \dot{x}(t)=-\infty$ where T is a positive number. In the remaining paragraphs other questions concerning (*) are studied, e.g. the existence of periodic solutions. *M. Zlámal* (Brno)

5782:

Olech, Czeslaw. Sur un problème de M. G. Sansone lié à la théorie du synchrotrone. Ann. Mat. Pura Appl. (4) **44** (1957), 317–329.

A partial solution is obtained for the following problem which was posed by G. Sansone [review above], and which is a generalization of the problem of determining the orbit of a particle in a synchrotron. Consider the equation (*) $\ddot{x}+x\phi(x)=p(t)$ where there are several conditions on $\phi(x)$ and $p(t)$ which will not be stated here. A solution of type (A) of (*) is a solution $x(t)$ defined as follows: let t^* be a fixed real number and let τ, c be positive numbers. Define sequence $\{t_i\}$ by: $t_i=t^*+i\tau$ $(i=0, \pm 1, \pm 2, \cdots)$. Corresponding to the pair of real numbers (ξ, η), define the function $x(t)$ by: $x(t)=x_0(t)$ for $t_0 \leqq t < \min(t_1, t_0^*)$ where $x_0(t)$ is a solution of (*) with initial conditions $x_0(t_0)=\xi$, $\dot{x}_0(t_0)=\eta$ and t_0^* is the right-hand endpoint of the maximum interval of existence of $x_0(t)$. If $x(t)$ is defined in $[t_0, t_i)$ and if the finite limits, $\lim_{t\to t_i-0} x(t)$ and $\lim_{t\to t_i-0} \dot{x}(t)$ exist, then let $\xi_i=\lim_{t\to t_i-0} x(t)+c \lim_{t\to t_i-0} \dot{x}(t)$ and let $\eta=\lim_{t\to t_i-0} \dot{x}(t)$. Then set $x(t)=x_i(t)$ for $t_i \leqq t < \min(t_{i+1}, t_i^*)$ where $x_i(t)$ is the solution of (*) determined by initial conditions $x_i(t_i)=\xi_i$, $\dot{x}_i(t_i)=\eta_i$ and t_i^* is the right-hand endpoint of the maximum interval of existence of $x_i(t)$. If $x(t)$ is defined for all $t \geqq t_0$, it is called a solution of type (A) of (*). The problem is to determine for given t^*, τ, c the set R of points (ξ, η) for which solutions of type (A) exist and are bounded in $[t_0, \infty)$. By making a discontinuous transformation on (*), the problem is changed to that of finding continuous solutions of a differential equation in which there appears a function with a denumerable set of discontinuities. The existence of a certain subset of R is then established. *J. Cronin* (New York, N.Y.)

5783:

Munteanu, Ion. Solutions bornées et solutions périodiques pour certains systèmes d'équations différentielles. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **8** (1957), 125–131. (Romanian. Russian and French summaries)

The author considers second order systems of the form (1) $\dot{x}=h(y)$, $\dot{y}=-f(x,y)y-g(x)+e(t)$, where $f(x,y)$, $g(x)$, $h(y)$, $e(t)$ are continuous functions of their arguments and $e(t)$ is bounded, $-\infty<x,y,t<+\infty$, and gives sufficient conditions in order that all solutions of (1) are ultimately bounded. I. If there is an $a>0$ such that $g(x)/x\geqq 1$ for all $|x|\geqq a$, and $f(x,y)|y|\to+\infty$ for $x^2+y^2\to\infty$ with $|y|\geqq a$; if there is an $M>0$ such that $f(x,y)>-M$ for $x>a$, $0\leqq y\leqq a$, and for $x<-a$, $-a\leqq y\leqq 0$; if sign $h(y)=$ sign y and $\int_0^\infty h(y)dy=\infty$; then there is a constant k, depending on the functions f, g, h, e only, such that for every solution $x(t)$, $y(t)$ of (1) there is some $\tau>0$ such that $|x(t)|$, $|y(t)|\leqq k$ for $t\geqq\tau$. If $e(t+\omega)=e(t)$ for some $\omega>0$ and (1) satisfies a uniqueness condition, then (1) has at least one periodic solution of period ω. This theorem, for which the idea of the proof can be traced in works of N. Levinson, improves a previous paper of I. Barbalat [Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **5** (1953), 393–402; MR **17**, 38] concerning the case $h(y)\equiv y$. *L. Cesari* (Lafayette, Ind.)

5784:

Seifert, George. A note on periodic solutions of second order differential equations without damping. Proc. Amer. Math. Soc. **10** (1959), 396–398; errata, 1000.

It is shown that the differential equation $\ddot{x}+g(x)=p(t)$ has a periodic solution of period T, under the assumption that $p(t)$ has period T, $g(x)$ is continuous and increasing, with $g|0|=0$, and $|g(x)|>\sup p(t)$ for $|x|$ sufficiently large, and provided that T is sufficiently small. The proof, which relies on a fixed-point theorem, is not given in full detail.

W. Kaplan (Ann Arbor, Mich.)

5785:

Loud, W. S. Periodic solutions of $x''+cx'+g(x)=\varepsilon f(t)$. Mem. Amer. Math. Soc. no. **31**, 58 pp. (1959).

In the following, when we say a function is periodic of a given period, we mean the least period. Consider the second order equation

$$(1)_{c,\varepsilon} \qquad x''+cx'+g(x)=\varepsilon f(t),$$

where ε, c are real constants, $f(t+T)=f(t)$ is piecewise continuous, $g(x)$, $g'(x)$ are continuous and $g''(x)$ is piecewise continuous, $xg(x)\geqq 0$, and the set of values x for which $g(x)=0$ is either $x=0$ or a closed interval containing $x=0$. Each solution of $(1)_{0,0}$ is periodic; for the nonconstant periodic solutions, let $T(A)$ be the period, where A is the positive amplitude and let $T'(A)=dT(A)/dA$. The author gives an explicit expression for $T'(A)$.

The purpose of the present paper is to discuss periodic solutions of $(1)_{c,\varepsilon}$ when ε is small and c is small and chosen after ε. More specifically, the author is interested in solutions of $(1)_{c,\varepsilon}$ which reduce for $\varepsilon=0$, $c=0$ to a periodic solution of $(1)_{0,0}$ which have a period in common with $f(t)$. First, for ε small, and using a procedure analogous to that of Poincaré, sufficient conditions are obtained for the existence of periodic solutions of $(1)_{0,\varepsilon}$. Then, for $c\neq 0$, the standard perturbation theory [see E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York-Toronto-London, 1955; MR **16**, 1022; ch. 14] can be applied to obtain periodic solutions of $(1)_{c,\varepsilon}$. Some, but not all of the results are stated below, and some parts of the theorem below are valid under weaker smoothness conditions on $g(x)$.

Theorem: Let $x_0(t)$ be a nonconstant periodic solution of $(1)_{0,0}$ having period $T(A_0)=rT$, where r is rational, and let L be the least common multiple of T and $T(A_0)$. (i) If $T'(A_0)\neq 0$, (2) $\int_0^L x_0'(t)f(t)dt=0$, (3) $M \int_0^L x''(t)f(t)dt\neq 0$, then there exists an $\varepsilon_1>0$, sufficiently small, such that there is a unique periodic solution $x(t,\varepsilon)$, $0\leqq|\varepsilon|\leqq\varepsilon_1$, of $(1)_{0,\varepsilon}$ having period L and such that $x(t,0)=x_0(t)$. (ii) For a fixed ε, $0<\varepsilon\leqq\varepsilon_1$, there exists a unique solution $x(t,\varepsilon,c)$ of $(1)_{c,\varepsilon}$ of period L such that $x(t,\varepsilon,0)=x(t,\varepsilon)$, provided that c is sufficiently small. (iii) This solution $x(t,\varepsilon,c)$ is asymptotically stable [conditionally stable] if $c>0$ and $-\varepsilon T'(A_0)M<0$ $[-\varepsilon T'(A_0)M>0]$. An interesting consequence of this theorem is the fact that it is possible to have more than one periodic solution of $(1)_{c,\varepsilon}$ if c is sufficiently small, as compared to ε. This stands in contrast to the facts for large ε and c. In particular, M. L. Cartwright and J. E. Littlewood [Ann. of Math. (2) **48** (1947), 472–494; MR **9**, 35] have shown that if c is large and positive, there is a unique periodic solution of $(1)_{c,\varepsilon}$ to which all others converge; and W. S. Loud [Duke Math. J. **24** (1957), 235–247, MR **19**, 36] has shown that if ε is large and $g(x)$ differs from a linear function by a bounded function, there tends to be just one periodic solution of $(1)_{c,\varepsilon}$. The author also discusses the case where condition (3) is not satisfied, but the results are too complicated to state here.

The above theorem deals with the case where $x_0(t)$ is nonconstant. Some results are also given in this paper concerning the existence and stability of periodic solutions of $(1)_{c,\varepsilon}$ in a neighborhood of $x=0$ for the case where $g(x)=0$ implies $x=0$, but the derivative of $g(x)$ at $x=0$ may or may not vanish. The paper concludes with a few applications. Related papers: A. M. Kac [Prikl. Mat. Meh. **19** (1955), 13–32; MR **16**, 822], Yu. M. Drozdov [ibid. **19** (1955), 33–40; MR **16**, 822], M. E. Levenson [J. Appl. Phys. **20** (1949), 1045–1051; MR **11**, 439].

J. K. Hale (Baltimore, Md.)

5786:

Loud, W. S. Periodic solutions of a perturbed autonomous system. Ann. of Math. (2) **70** (1959), 490–529.

Consider the differential system

$$(1) \qquad x'=g(x)+\varepsilon f(t,x,\varepsilon),$$

where ε is a real parameter, x is an n-vector with all components real-valued functions of the real variable t, and $g(x)$, $f(t,x,\varepsilon)$ are sufficiently smooth functions of t, x, ε (the author never requires more and sometimes requires less than three derivatives with respect to x and two with respect to t, ε). Also, $f(t,x,\varepsilon)$ is periodic in t of least period T. Suppose that the equation (2) $x'=g(x)$ has a periodic solution $x=x_0(t)$ of period L_0 and that the equation (3) $y'=g'(x_0(t))y$ has only one characteristic multiplier equal to $+1$. The author obtains sufficient conditions (which are too complicated to state here) for the existence and stability of periodic solutions of (1) in the case where T, L_0 have rational ratio. To prove the results, he uses a perturbation technique very similar to the one of his previous paper [review above] for a single second order equation. However, the results do not overlap. The results are then applied to van der Pol's equation $u''+\mu(u^2-1)u'$

$+u=h(t)$ with μ not necessarily a small parameter. Theorems 1, 3 and 5 of the paper under review are very closely related to some results of the book of I. G. Malkin [*Nekotorye zadači teorii nelineĭnyh kolebaniĭ*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **18**, 396; Ch. 6]. [Related papers: N. Levinson, Ann. of Math. (2) **52** (1950), 727–738; MR **12**, 335; G. Hufford, *Contributions to the theory of nonlinear oscillations*, vol. 3, Princeton Univ. Press, 1956; MR **18**, 654; pp. 173–195.]

J. K. Hale (Baltimore, Md.)

5787:

Halanay, A. Sur un critère de stabilité. Com. Acad. R. P. Romîne **9** (1959), 209–214. (Romanian. Russian and French summaries)

The author considers a 2-dimensional system $dy/dt = Y(y, t)$ with Y analytic in y and bounded in t for $t \geqq 0$ which has a family of solutions $\phi(t, h)$, depending analytically on h, such that every $\phi(t, h)$ and every $\partial^k\phi/\partial h^k(t, 0)$ is bounded for $t \geqq 0$. He proves that $\phi(t, 0)$ is uniformly stable provided $\|\partial\phi/\partial h(t, 0)\| \geqq \gamma > 0$ and

$$\int_{t_0}^{t} \operatorname{trace}\left(\frac{\partial Y}{\partial y}[\phi(t, 0), t]\right)dt \leqq -\nu(t-t_0)+\chi(t),$$

where $\nu > 0$ and $\chi(t)$ is bounded for $t \geqq 0$.

H. A. Antosiewicz (Los Angeles, Calif.)

5788:

Halanaĭ, A. Stability theorems for a system with lagging argument. Rev. Math. Pures Appl. **3** (1958), 207–215. (Russian)

The following theorems are proved. 1. Let $f(t, x, y)$ be Lipschitzian in $x, y, f(t, 0, 0)=0$; if the trivial solution of

$$\dot{x}(t) = f(t, x(t), x(t-\tau)) \tag{1}$$

is uniformly asymptotically stable (uas), then it is totally stable (i.e., under constantly acting perturbations). 2. Under the same assumptions, if $R(t, x, y)$ is also Lipschitzian in x, y, $R(t, 0, 0)=0$, $R(t, x, y)\to 0$ when $t\to+\infty$, uniformly in x, y, the trivial solution of

$$\dot{x}(t) = f(t, x(t), x(t-\tau))+R(t, x(t), x(t-\tau))$$

is also uas. 3. Let

$$\begin{aligned}\dot{x}(t) &= X(t, x(t), x(t-\tau), y(t), y(t-\tau)),\\ \dot{y}(t) &= Y(t, x(t), x(t-\tau), y(t), y(t-\tau)),\end{aligned} \tag{2}$$

where $X(t, 0, 0, 0, 0)= Y(t, 0, 0, 0, 0)=0$ and assume that the trivial solution of (2) is (asymptotically) stable with respect to x (i.e., x remains arbitrarily small in the future provided that the initial conditions on x, y are sufficiently small); assume furthermore that the trivial solution of the second equation alone, with x replaced by 0, is uas; then the trivial solution of (2) is uniformly stable [uas].

J. L. Massera (Montevideo)

5789:

Yoshizawa, Taro. On the equiasymptotic stability in the large. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1959), 171–180.

The author continues his studies [same Mem. **29** (1955), 275–291; MR **20** #4679] of stability for the differential system (1) $dx/dt = F(t, x)$, where x is a real n-vector, $F(t, x)$ is continuous in $0 \leqq t < \infty$ and all x, and $F(t, 0)\equiv 0$. The solution $x(t)\equiv 0$ of (1) is said to be equi-asymptotically stable in the large if there exists a positive constant $T(t_0, \alpha, \varepsilon)$, defined for $\varepsilon > 0$, $\alpha > 0$, $t_0 \geqq 0$, such that $\|x_0\| \leqq \alpha$ and $t > t_0 + T(t_0, \alpha, \varepsilon)$ imply $\|x(t; x_0, t_0)\| < \varepsilon$.

The main result is that a necessary and sufficient condition for the solution $x(t)\equiv 0$ of (1) to be equi-asymptotically stable in the large is the existence of a continuous Lyapunov function $\phi(t, x)$ satisfying certain growth conditions.

L. Markus (Minneapolis, Minn.)

5790:

Karaseva, T. M. Boundedness test and the exact evaluation of the multipliers of solutions to Hill's equations. Dokl. Akad. Nauk SSSR **127** (1959), 1161–1163. (Russian)

Let Q_α be the family of real square integrable functions $q(x)$, periodic with period T and mean value zero, such that $T\int_0^T q^2(x)dx=\alpha$; let $P_{\alpha\gamma}$ be the family of functions $p(x)=q'(x)+\gamma$ with $\gamma\geqq 0$ and $q\in Q_\alpha$; and let $y''+p(x)y=0$ with $p\in P_{\alpha\gamma}$. If all solutions are bounded, then (γ, α) is said to belong to the stability region, which is known to contain the segment $\gamma=0$, $0<\alpha<\pi^2/4$ and all points with $\gamma\geqq 0$, $\alpha<2-T^4\gamma^2/24$.

The author constructs in the first quadrant of a $(T\sqrt{\gamma}, \sqrt{\alpha})$-plane closed regions D_n whose boundaries are the $\sqrt{\alpha}$-axis on the left and curves Γ_n, Γ_n' below and to the right, given parametrically, where Γ_n [Γ_n'] is convex [concave] and Γ_n passes through $(0, n\pi/2)$ and intersects Γ_n' at $(n\pi, 0)$. The complement of $\bigcup D_n$ in the first quadrant is asserted to be the stability region.

H. A. Antosiewicz (Los Angeles, Calif.)

5791:

★**Зубов, В. И. Математические методы исследования систем автоматического регулирования. [Zubov, V. I. Mathematical methods for investigation of control systems.]** Gosudarstv. Sudprom. Izdat. Leningrad, 1959. 224 pp.

The present book is one more noteworthy addition to the long list of Soviet books on the problems related to Liapunov's classical direct method and its applications. Its title is in a certain sense misleading since automatic controls hardly occupy more than one sixth of the work (mostly the last chapter). In reality the book deals largely with the "Liapunov" problems. Thus its subject-matter overlaps to quite an extent that of the recently reviewed book by Krassovskiĭ [*Nekotorye zadači teorii ustoĭčivosti dviženiya*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1959; MR **21** #5047]. It overlaps but by no means coincides with it. Indeed Zubov's book is fascinating reading and abounds in original results. The author is also more definitely aware of practical applications than most Soviet authors, who in this regard are often satisfied with a vague "very important for practical applications" without telling why. In short, we have here a highly interesting production covering a very wide range. We must protest however against the poor proofreading and typography, sometimes bad choice of notations, and (very Soviet!) highly restricted use of the vector-matrix notation making at times for rather rocky reading, especially since the author is very strongly inclined to analysis.

A chapter by chapter description follows.

Chapter I. General stability theorems. The basic system is a non-autonomous n-vector system (1) $\dot{x}=f(x, t)$, $f(0, t)=0$, $0\leqq t<\infty$, x over the whole space. The basic definitions are recalled and clarified by examples. A

number of results of necessary-and-sufficient type are then given; they include Liapunov's theorems and their converses. A noteworthy definition: Let $x(t)$ denote here and later the solution such that $x(t_0)=x_0$, $t_0 \geqq 0$. Then the origin is said to be uniformly attractive whenever (a) it is asymptotically stable and (b) to any $h>0$ there correspond α, $T>0$ such that if $\|x_0\|>h$ then $\|x(t)\|>\alpha$ for $t \in [t_0, t_0+T]$. A necessary and sufficient condition (n.a.s.c.) for the possession of this property is given. Sufficient conditions are: (a) existence of a positive definite function $V(x, t)$ (in the sense of Liapunov) dominated by a similar $W(x)$, such that $W(0)=0$; (b) along the trajectories $\dot{V} \leqq 0$; (c) there exists a function $\psi_\beta(t)$, $\psi_\beta \leqq \dot{V} \leqq 0$ for $\|x\|>\beta$, such that $\int_{t_0}^{t_0+T} \psi_\beta(\tau)d\tau \to 0$ with T uniformly relative to t_0. Questions such as these are also discussed: stability relative to certain coordinates, size of the domain of asymptotic stability.

Chapters II, III. Estimate of the magnitude of transient processes in linear systems (Ch. II) and nonlinear systems (Ch. III). The linear systems are (2) $\dot{x}=P(t)x$, and they are more or less thought of as approximations to (1). As is well known, when P is not constant the treatment is not at all the same as when it is, and it causes many difficulties. A wealth of results is given relating the behavior of the functions V, $\dot{V}$ to certain restrictions on solutions of (2). Ch. III deals first with a rather involved concept of technical stability (on a finite interval $[t_0, t_0+T]$) and complicated inequalities relating the solutions of (1) on the interval to the behavior of a suitable $V(x, t)$. The author also gives his definition of stability on $[t_0, t_0+T]$ relative to a function $V(x, t)$ which is positive definite in the sense of Liapunov: given $a>0$, the assumption is that if $V(x_0, t_0)=a$ then $V(x(t), t)<a$ for t in the interval, whatever a. Here $x(t)$ is the same solution of (1) as before. Take (3) $\dot{x}=P(t)x+f(x, t)$. Suppose that $f(x, t)$ is known for $t \geqq 0$ and $\|x\| \leqq \alpha$, $\alpha>0$. Let $\varphi(t)$ be a continuous positive function, defined for $t \geqq 0$ and let $c>1$. Let finally (4) $\|f\|<\varphi(t)\|x\|^c$. Then: (Theorem) In order that (3) be stable on some $[t_0, t_0+T]$ relative to a fixed positive definite quadratic form V, and this whatever f satisfying (4), it is n.a.s. that the characteristic roots of $P(t_0)$ all have negative real parts. A brief summary of various results, notably some due to Bellman on the stability of differential equations with retardation, concludes Chapter III.

Chapter IV. Solutions of analytic systems in the neighborhood of a singular point. Various developments along the line of explicit series solutions such as considered in Liapunov's so-called first method.

Chapter V. Problems of persistent disturbances. Effect on the n-vector system of a persistent disturbance $r(x, t)$ represented by (5) $\dot{x}=f+r$. Malkin's theorem is re-proved to the effect that if (1) is uniformly asymptotically stable then (5) is stable for r small enough. Various inequalities are given for $\|x(t)\|$ and a finite interval $[t_0, t_0+T]$. The stochastic case (random vectors f, r) is likewise discussed.

Chapter VI. Critical cases. Suppose that in (1) the linear terms have a constant coefficient matrix with some characteristic roots with zero real parts. This is known as a critical case. Let the system be of type (6a) $\dot{y}=Y(y, z, t)$; (6b) $\dot{z}=Z(y, z, t)$, $t \geqq 0$, where y is a p-vector, and z is a q-vector. Assume further that $Y=0$ for $y=0$ and $Z=0$ for $y=0$, $z=0$. We operate in a domain $\|y\|$, $\|z\|<H$. The author does not deal symmetrically with the y, z spaces. Take the system (7) $\dot{y}^*=Y(y^*, \zeta(t), t)$, where $\|\zeta(t)\| \leqq H$ for $t \geqq 0$. The origin $y^*=0$ is called strongly stable for (7) if the system is stable in the ordinary sense for all $\|\zeta\|<H_1$. If moreover $y^*(t) \to 0$ as $t \to +\infty$ then (7) is strongly asymptotically stable. The function $W(y, z, t)$ is said to be strongly definite negative under the following circumstances: there exists a vector function $\varphi(z)$, with $\varphi(0)=0$, and all components of $\varphi(z)$ positive otherwise, also a function $W(y, z, t)$ which is definite negative for all choices of $y(t)$ such that the components of $y-\varphi$ are all non-positive. Theorem: Suppose that (a) the origin is strongly stable [asymptotically stable] for (7); (b) there exists a continuously differentiable function $V(z, t)$ which is uniformly continuous as to t for t large near $z=0$; (c) the function $W(y, z, t)=\partial V/\partial t+Z \cdot \operatorname{grad} V$ is strongly definite negative in z. Then the origin for (6) is also stable [asymptotically stable]. Several further theorems are also given in the chapter.

An extensive study is also made of the autonomous systems (8) $\dot{x}=X(x)$ which are homogeneous, and this to a variable degree, in the components of x. Theorem: N.a.s.c. for the asymptotic stability of (8) is the existence of two homogeneous functions V, W of suitable (specified) degree with $W=\dot{V}$ and V, $-W$ positive definite in the whole space. This and related results are then exploited for non-homogeneous systems. Application is made to a type of question dealt with by Malkin: analytic systems where the stability depends upon the terms of degree $\leqq N$ in y regardless of the terms of higher degree in that variable. One assumes that upon making $y=0$ the system is stable or asymptotically stable.

Chapter VII. Periodic and almost periodic solutions. A rather complete discussion of periodic solutions of linear systems. Discussion of the region of attraction of a closed trajectory in an autonomous system. Proof that if $\dot{x}=X(x)$ has a periodic solution so has $\dot{x}=X(x)+R(x)$ one nearby under certain conditions. Forced oscillations of linear systems: such systems have a unique periodic solution asymptotically stable in the large. N.a.s.c. for this to hold for $\dot{x}=X(x)+f(t)$. Study of the possible periodic and almost periodic solutions of analytic systems with the origin as critical point and a certain number of pairs of pure complex characteristic numbers. Discussion of approximations à la van der Pol (ignored by the author) and Krylov-Bogoliubov.

Chapter VIII. Application of high-speed calculating machines to automatic controls. It is above all in this chapter of about 40 pages (and in a couple of appendices) that these questions are discussed. The discussion is very extensive, even bearing upon systems with retardation. There is a noteworthy description (12 pages) of high-speed computers by Chernenko.

S. Lefschetz (Mexico City, D.F.)

5792:

Barocio, Samuel. Singularities of analytic differential systems in the plane. Bol. Soc. Mat. Mexicana (2) 4 (1959), 1–25. (Spanish)

The author discusses the structure of the trajectories of the system (1) $\dot{x}=X(x, y)$, $\dot{y}=Y(x, y)$, near an isolated common zero of $X(x, y)$ and $Y(x, y)$. $X(x, y)$ and $Y(x, y)$ are assumed to be holomorphic in a neighborhood D of the isolated singularity (taken to be the origin), $X=X_p+\cdots$, $Y=Y_q+\cdots$, where X_r, Y_r are homogeneous of degree r. The arcs through the origin of the places of $X=0$, $Y=0$

divide D into subregions in each of which X and Y have fixed sign. These arcs together with the "curves of access", the trajectories of (1) whose limit sets contain the origin, play a key role in the author's analysis. By the Weierstrass and the Puiseux factorization theorems [Goursat, *Mathematical Analysis II*, part 1, Ginn, Boston, 1904] (1) is transformed into

$$(2)\qquad \begin{aligned} \dot{x} &= \alpha F(x, y) \prod (y - x^{p_i/m_i} E_i(x^{1/m_i}))^{\sigma_i}, \\ \dot{y} &= \beta G(x, y) \prod (y - x^{q_j/n_j} E_j(x^{1/n_j}))^{\tau_j}, \end{aligned}$$

where $\alpha\beta \neq 0$, F and G vanish at the origin but are positive elsewhere in D, and $E_i(0) = E_j(0) = 1$. Using (2) the author gives a complete discussion, with one exception, of the topological structure of the trajectories in the various subregions of D. The exception is known as the fan problem and has been treated by Lefschetz [Bol. Soc. Mat. Mexicana (2) **1** (1956), 13–27; **2** (1957), 63–74; MR **18**, 481; **20** #1320]. The spirit throughout the paper is geometric; the methods and results are closely connected with the work of Lefschetz [ibid., and *Differential equations: geometric theory*, Interscience, New York, 1957; MR **20** #1005; ch. X]. It should be noted that the general problem of determining the structure of trajectories near a singularity is quite old. Pertinent references include: Briot and Bouquet [J. École Poly. **36** (1856), 133–198], Poincaré [*Œuvres*, 1, Gauthier-Villars, Paris, 1928; pp. 90–222], Bendixson [Acta Math. **24** (1901), 1–88], Frommer [Math. Ann. **99** (1928), 222–272], and Forster [Math. Z. **43** (1937–38), 271–320]. *C. S. Coleman* (Claremont, Calif.)

5793:

Čečik, V. A. Investigation of systems of ordinary differential equations with a singularity. Trudy Moskov. Mat. Obšč. **8** (1959), 155–198. (Russian)

The present paper concerns the difficult question of existence, uniqueness, continuous dependence upon parameters, and approximation, of solutions of real differential systems in a neighborhood of a singular point. Systems of the form

$$(1)\qquad dy_k/dx = f_k(x, y_1, \cdots, y_n) \quad (k = 1, \cdots, n)$$

are investigated, where $x = 0$ is the singular point, and the real functions f_k are supposed to be continuous in their arguments, together with their first order partial derivatives f'_{k,y_i} $(i, k = 1, \cdots, n)$ in a region D of the form $0 < x \leq b$, $|y_i| \leq a$ $(i = 1, \cdots, n)$. Solutions $y_k(x)$ $(k = 1, \cdots, n)$ are sought satisfying system (1) in some neighborhood of $x = 0$, $0 < x \leq c$ $(0 < c \leq b)$, and the initial condition

$$(2)\qquad y_k(0+) = 0 \quad (k = 1, \cdots, n).$$

The following is one of the existence theorems proved by the author. (I) If

$$|f_k(x, y_1, \cdots, y_{k-1}, 0, y_{k+1}, \cdots, y_n)| \leq \psi(x) \quad (k = 1, \cdots, n),$$

$$|f'_{k,y_k}(x, y_1, \cdots, y_n)| \leq \psi(x) \quad (k = 1, \cdots, p),$$

$$f'_{k,y_k}(x, y_1, \cdots, y_n) \geq \bar{\psi}(x) \quad (k = p+1, \cdots, n, \text{ for some } p,\ 0 \leq p \leq n),$$

where $\psi(x)$, $\bar{\psi}(x)$ are non-negative continuous functions in $(0, b]$, ψ summable and $\bar{\psi}$ nonsummable in $(0, b]$, then (1) has at least one solution $y(x)$ satisfying (2). In addition, the author proves that, for $p < n$, there is at least an $(n-p)$-parameter family of solutions $y(x)$ satisfying (2). For $n = 1$, the following negative theorem is proved. (II) If there is a continuously differentiable function $\bar{\phi}(x)$, $0 \leq x \leq b$, $\bar{\phi}(0) = 0$, such that $f_y'(x, y) > 0$ for $y > \bar{\phi}(x)$, $f(x, y) \geq \max \bar{\phi}'(x)$, where max is taken for $0 \leq x \leq b$, and $\int_c^b f(x, \bar{\phi}(x))dx = +\infty$, then the equation $y' = f(x, y)$ has no solution $y(x)$ with $y(0+) = 0$. For $n \geq 1$, $p = n$, theorem I does not assure uniqueness as is shown by examples. A uniqueness theorem was given by M. A. Krasnosel'skiĭ and S. G. Kreĭn [Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal. no. 2 (1956), 3–23; MR **19**, 140] slightly improving a previous analogous result of L. Tonelli [Rend. Accad. Naz. Lincei (6), **1** (1925), 272–277]. The author proves the following uniqueness theorem. (III) If

$$|f_k(x, y_1, \cdots, y_{k-1}, 0, y_{k+1}, \cdots, y_n)| \leq \psi(x) \quad (k = 1, \cdots, n),$$

$$f'_{k,y_k}(x, y_1, \cdots, y_n) \leq \psi(x) \quad (k = 1, \cdots, n),$$

$$|f'_{k,y_i}(x, y_1, \cdots, y_n)| \leq \psi(x) \quad (i \neq k;\ i, k = 1, \cdots, n),$$

everywhere in D, where $\psi(x)$ is a non-negative function in $(0, b]$, summable in $(0, b]$, then (1) has one and only one solution $y(x)$ satisfying (2). Under the conditions of III, analogous theorems assure continuous dependence upon parameters. Finally, a method of successive approximations is proposed, and its convergence proved. Also, evaluations of the error after the mth approximation are given. Note that for the nth order differential equation

$$(3)\qquad y^{(n)} = f(x, y', \cdots, y^{(n-1)})$$

the following statement holds. (IV) If the functions $|f(x, y_1, \cdots, y_{n-1}, 0)|$, $|f_{y_k}'(x, y_1, \cdots, y_n)|$ $(k = 1, \cdots, n-1)$, $f_{y_n}'(x, y_1, \cdots, y_n)$ are all $\leq$ a given non-negative continuous function $\psi(x)$, summable in $(0, b]$, then there is one and only one solution $y(x)$ of (3) satisfying $y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0$. *L. Cesari* (Lafayette, Ind.)

5794:

Peixoto, Marilia C.; and Peixoto, M. M. Structural stability in the plane with enlarged boundary conditions. An. Acad. Brasil. Ci. **31** (1959), 135–160.

Consider the class of differential equations (X) $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ with P, Q in C^1 in a neighborhood of G, a closed bounded plane region with a Jordan boundary curve L of class C^1. Using the C^1-norm in G, we find that the set of all such differential systems is a Banach space $\mathscr{B}$. A differential system X in $\mathscr{B}$ satisfies the boundary condition B in case: (1) X has no critical point on L and no limit cycle tangent to L from inside G; (2) X has at most a finite number of points of tangency with L, and at each such point L the trajectories of X have distinct curvatures; (3) each trajectory of X which meets L is tangent to L at most once; and (4) no trajectory of X connecting L to a saddle point of X is tangent to L.

A differential system X in $\mathscr{B}$ is called structurally stable (in the wide sense) in case X satisfies B and also X is topologically equivalent to all neighboring members of $\mathscr{B}$.

The author proves that the set of structurally stable differential systems is open and dense in the metric space $\mathscr{B}$. The proof follows a difficult construction which shows that X, satisfying B, is structurally stable if and only if the Andronov-Pontrjagin conditions are fulfilled.

L. Markus (Minneapolis, Minn.)

5795:

Loewner, Charles. A theorem on the partial order derived from a certain transformation semigroup. Math. Z. **72** (1959/60), 53–60.

The author determines the closure S of the union of all trajectories $u(t)$, $t \geqq a$, in real $n+1$ space which are solutions of differential equations of the form $du/dt = A(t)u$, where $u(a) = (1, 0, \cdots, 0)$ and $A(t) = (a_{ij}(t))$ is a matrix of order $n+1$ which is integrable in every finite interval and which satisfies the following conditions: (a) $A(t)$ is symmetric for each t; (b) $a_{ij}(t) \geqq 0$ almost everywhere for $i \neq j$; and (c) $\sum_j a_{ij}(t) = 0$ almost everywhere for each i. The set S turns out to be a convex set, namely the set of all points $(u_0, \cdots, u_n)$ such that $\sum_{i=0}^n u_i = 1$, and $(u_1, \cdots, u_n) \in B$, where B is the polyhedron whose vertices are the points $P_0 = (0, \cdots, 0)$, $P_k = (1/2, 1/2^2, \cdots, 1/2^k, 0, \cdots, 0)$, $k = 1, \cdots, n$, together with all points whose coordinates are a permutation of those of one of the P_k. It is stated that this problem is the finite-dimensional analogue of a problem in the theory of linear parabolic partial differential equations. Remarks: Lemmas 1 and 2 are well known from the theory of doubly stochastic matrices. The set B may be described in other ways, for example, by the inequalities $0 \leqq u_{i_1} + \cdots + u_{i_k} \leqq 1/2 + \cdots + 1/2^k$, for $1 \leqq i_1 < \cdots < i_k \leqq n$, $1 \leqq k \leqq n$.

A. Horn (Los Angeles, Calif.)

5796:

Bellman, Richard. Asymptotic series for the solutions of linear differential-difference equations. Rend. Circ. Mat. Palermo (2) **7** (1958), 261–269.

It is shown that the difference-differential equation

$$u'(t) = a(t)u(t) + b(t)u(t-1), \tag{1}$$

where

$$a(t) \sim a_0 + a_1/t + \cdots, \quad b(t) \sim b_0 + b_1/t + \cdots,$$

as $t \to \infty$, has the formally expected asymptotic solution. The method of proof is to transform (1) by $u(t) = e^{a_0 t} t^r v(t)$, with r determined so that

$$v'(t) - b_0 v(t-1) = F(t), \tag{2}$$

where $F(t)$ is a linear expression in $v(t)$ and $v(t-1)$ with coefficients $O(1/t)$ as $t \to \infty$. The theory of linear difference-differential equations with constant coefficients then gives $v(t)$ as an integral involving $F(t)$. Its $O(1/t)$ coefficients permit the determination of the order of $v(t)$ after some calculation.

The method can readily be extended to more complicated systems at the cost of more algebra.

E. Pinney (Berkeley, Calif.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 5845, 5873, 5874, 5901, 5992, 6005, 6100, 6167, 6231.

5797:

Szarski, J.; et Ważewski, T. Interprétation géométrique des conditions d'intégrabilité d'un système d'équations aux différentielles totales. Ann. Polon. Math. **6** (1959), 301–304.

The authors consider a system of equations of the type:

$$dz^i = P^i(x, y, z^j)dx + Q^i(x, y, z^j)dy \quad (i, j = 1, \cdots, n),$$

the functions P^i and Q^i being of class C^1 in a domain Ω. A geometrical interpretation (too complicated for reproduction in a review) of the well-known integrability conditions of this system is derived by means of the intersections of certain surfaces in the (x, y, z^i)-space.

H. Rund (Durban)

5798:

Iordanskiĭ, S. V. Solution of the Cauchy problem for the kinetic equation of an electron plasma. Dokl. Akad. Nauk SSSR **127** (1959), 509–512. (Russian)

The problem posed is to find functions $n(v, x, t)$ and $E(x, t)$ that satisfy the system

$$\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} - \frac{e}{m}E(x, t)\frac{\partial n}{\partial v} = 0,$$

$$\frac{\partial E}{\partial x} + 4\pi e\left\{\int_{-\infty}^{\infty} n(v, x, t)dv - N_0\right\} = 0,$$

with the boundary conditions $n(v, x, 0) = f(x, v)$, $\lim_{x\to\pm\infty} E(x, t) = 0$ and $\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} f(x, v)dv - N_0\}dx = 0$. When a number of regularity conditions have been imposed, the problem reduces to the solution of a certain non-linear integral equation for the function E. Existence and uniqueness of the solution of the original problem are proved from the properties of the integral equation.

R. N. Goss (San Diego, Calif.)

5799:

Janet, Maurice. Sur la classification des systèmes formés d'autant d'expressions différentielles linéaires indépendantes que de fonctions indéterminées. J. Math. Pures Appl. (9) **37** (1958), 279–293.

This is a classification of systems of partial differential equations by means of the rank of the determinant of the system. The second and third order cases are treated in detail.

J. A. Ward (Alamogordo, N.M.)

5800:

Lebedev, V. I. A finite-difference analog of the Neumann problem. Dokl. Akad. Nauk SSSR **126** (1959), 494–497. (Russian)

Let there exist a generalized solution $u(x_1, \cdots, x_n)$ in a region Ω (of boundary S) of the equation (1) $\Delta u = f$, satisfying (in the weak sense) the boundary condition (2) $\partial u/\partial n|_S = \varphi$, with $f \in L_2(\Omega)$, $\varphi \in L_2(S)$. Suppose $\int_\Omega f d\Omega = \int_S \varphi dS$, that Ω is star-shaped, and that $\int_\Omega u d\Omega = 0$. Divide n-dimensional space into a lattice of cubes $\Omega_{k_1\cdots k_n}$ by the planes $x_i = k_i h$ $(i = 1, \cdots, n)$ where the k_i are integers; and let Ω_h (of boundary S_h) be the region formed by those cubes that belong to Ω. Set $u_{x_i}(x_i + h/2) = h^{-1}(u(x_i + h) - u(x_i))$. At the interior lattice points, equation (1) is replaced by (5) $\Delta_h u = f_h$, where f_h is some approximation of f. Corresponding to a boundary lattice point x_0 $(\in S_h)$, if there are k interior lattice points $x_1, \cdots, x_k$ at a distance h from x_0 (x_0 is then a point of type k), set (6) $l_h(u) = h^{-1}(ku(x_0) - \sum_1^k u(x_j)) = \varphi_h$. Theorem: For given φ_h on S_h, system (5), (6) is soluble if and only if (7) $h^n \sum_{\Omega_h} f_h = h^{n-1} \sum_{S_h} \varphi_h$.

Two cases are discussed: (a) $\partial u/\partial n|_S = 0$; (b) $f = 0$.

In (a), f_h is chosen so that $h^n\sum_{\Omega_h} f_h=0$, and one takes $l_h(u)=0$ on S_h, and $h^n\sum_{\Omega_h} u=0$. It is then stated that the solution u_h of the approximate system converges weakly in $W_2^{(1)}$, as $h\to 0$, to a generalized solution of the boundary problem. Case (b) is discussed in some detail for $n=2$ and $n=3$, leading to a convenient iteration process for the solution. *I. M. Sheffer* (University Park, Pa.)

5801:

Il'in, V. A.; and Šišmarev, I. A. On the connection between the classical and the generalized solution to Dirichlet's problem and to the problem of eigenvalues. Dokl. Akad. Nauk SSSR **126** (1959), 1176–1179. (Russian)

After summarizing conditions for the existence of classical and generalized solutions to the Dirichlet problem for second order linear elliptic operators (and for the existence of complete orthonormal sets of eigenfunctions in both senses), the authors state results on when classical solutions are generalized solutions (for unbounded regions). *R. R. D. Kemp* (Kingston, Ont.)

5802:

Hartman, Philip. On solutions of $\Delta V+V=0$ in an exterior region. Math. Z. **71** (1959), 251–257.

Let $x=(x^1, \cdots, x^p)$ denote a point in the p-dim. Euclidean space, Δ be the Laplacian and $V=V(x)$ a solution (of class C^2) of (1) $\Delta V+V=0$ in the entire space. Herglotz has shown that under the boundedness condition (2) $r^{-1}\int_{|x|\leq r}|V(x)|^2dx<C$, for all r, certain "local means" of V satisfy an asymptotic relation, as $|x|\to\infty$. The author proves that in the L^2-average V itself satisfies the corresponding asymptotic relation: (I) If U denotes the surface of the unit sphere and V is an entire solution of (1) satisfying (2), then

$$r^{-1}\int_0^r \|t^{(p-1)/2}V(tu)-W(tu)\|^2dt\to 0,$$

as $r\to\infty$, where $\|\cdots\|$ denotes the L^2-norm on U, $W(ru)$ is defined by

$$W(ru) = t^{(p-1)/2}e^{tr}\cdot K(u)+t^{(p-1)/2}\cdot e^{-tr}\cdot K(-u)$$

and $K(u)$ is a function on U. From this theorem the author gets easily a result given by Magnus [Abh. Math. Sem. Univ. Hamburg **16** (1949), 77–94; MR **11**, 176]: If V satisfies the hypothesis of (I) and if the relation $r^{(p-1)/2}\cdot V(ru)\to 0$, as $r\to\infty$, holds for every (fixed) u belonging to a "half-sphere", then $V(x)\equiv 0$. The author gets similar results for functions V, which are not entire solutions of (1) but satisfy (1) only in the exterior of a sphere. *A. Pfluger* (Zürich)

5803:

Protter, M. H. A comparison theorem for elliptic equations. Proc. Amer. Math. Soc. **10** (1959), 296–299.

The author obtains a comparison theorem of the Sturm type for a pair of linear elliptic partial differential equations

$$\partial(a_{ij}^k\partial u/\partial x_j)/\partial x_i+b_i^k\partial u/\partial x_i+f^k u=0,$$

where $k=1, 2$. The assertion of such a theorem is to the effect that if D is a bounded domain with boundary Γ and if the first equation has a non-trivial solution $u\geqq 0$ on D vanishing on Γ, then every solution of the second on $D+\Gamma$ has a zero. This paper generalizes a result of the reviewer and Wintner [same Proc. **6** (1955), 862–865; MR **17**, 627] for the case $b_i^k=0$. The main conditions of the author's theorem are an inequality $(a_{ij}^2)^{-1}\geqq(a_{ij}^1)^{-1}$ in the sense of Hermitian matrices and another inequality, too complicated to state here, involving a_{ij}^k, b_i^k, f^k and $\partial b_i^k/\partial x_j$. In the case of two independent variables and $b_i^2=0$, this inequality reduces to $f^2\geqq f^1-\partial b_1^1/\partial x_1-\partial b_2^1/\partial x_2$. *P. Hartman* (Baltimore, Md.)

5804:

Magenes, Enrico. Recenti sviluppi nella teoria dei problemi misti per le equazioni lineari ellittiche. Rend. Sem. Mat. Fis. Milano **27** (1957), 75–95.

Sia Ω un insieme aperto e limitato di R^m e in Ω sia dato l'operatore differenziale del secondo ordine di tipo ellittico:

$$E(u) = \sum_{i,j}^{1\cdots m} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^m b_i(x)\frac{\partial u}{\partial x_i}+c(x)\cdot u.$$

Il problema misto consiste nel cercare una funzione u in Ω che soddisfa l'equazione: $E(u)=f$ e inoltre sulla frontiera $\partial\Omega$ di Ω soddisfa le condizioni seguenti: $u=\mu$ su $\partial_1\Omega$; $du/d\nu=\delta$ su $\partial_2\Omega$ essendo $\partial_1\Omega\cup\partial_2\Omega=\partial\Omega$ e $du/d\nu$ la derivata conormale relativa all'operatore $E(u)$.

Si tratta di una accurata esposizione di vari risultati relativi all'esistenza e alla regolarizzazione della soluzione del problema misto. Un'ampia bibliografia correda l'esposizione. *G. Stampacchia* (Genoa)

5805:

Simoda, Seturo. Sur la condition frontière dans le problème de Dirichlet pour les équations semi-linéaires du type elliptique et du second ordre. Proc. Japan Acad. **35** (1959), 115–119.

È noto che la nozione di regolarità o di irregolarità dei punti frontiera di un dominio rispetto al problema di Dirichlet per le equazioni lineari di tipo ellittico e del secondo ordine è indipendente dalla equazione stessa [cfr. W. Püschel, Math. Z. **34** (1932), 535–553; G. Tautz, Math. Nachr. **2** (1949), 279–303; MR **11**, 358; O. A. Oleĭnik, Mat. Sb. (N.S.) **24 (66)** (1949), 3–14; MR **10**, 713]. L'A. generalizza questo risultato considerando equazioni differenziali del secondo ordine di tipo ellittico semilineari del tipo:

$$\sum_{i,j=1}^m a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^m b_i(x)\frac{\partial u}{\partial x_i} = F\left(x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_m}\right).$$

G. Stampacchia (Genoa)

5806:

MacCamy, Richard C. Asymptotic developments for a boundary value problem containing a parameter. Quart. Appl. Math. **17** (1959), 155–163.

The purpose of this paper is to point out a curious type of singularity which can arise in the perturbation of the solutions of boundary value problems containing a parameter. The author considers the determination of a harmonic function in the half plane $y<0$ under the

following boundary conditions. Problem I: $u_y(x,0)=0$, $|x|>1$, $u_y(x,0)+Ku(x,0)=g(x)$, $|x|<1$; Problem II: $u_y(x,0)-Ku(x,0)=0$, $|x|>1$, $u_y(x,0)=g(x)$, $|x|<1$; where K is a positive constant, $g(x)$ a given function. The problems are identical when $K=0$.

As K tends to zero, the solution of problem I tends to the limit $\pi^{-1}\int_{-1}^{1} g(t)\log[(x-t)^2+y^2]dt$, which is the solution of the problem with $K=0$; but the solution of problem II does not tend to a limit as $K\to 0$.

The solution of problem I is, in fact, an analytic function of K, now regarded as a complex variable, regular in a neighbourhood $|K|\leq\rho$ of the origin; but the solution of problem II is not regular in such a neighbourhood, but only in a suitably cut annulus $\rho_0\leq|K|\leq\rho$, and has a singularity at $K=0$.

The two problems indicate the need for caution in the use of perturbation methods.

E. T. Copson (St. Andrews)

5807:

Pogorzelski, W. Problème aux limites aux dérivées tangentielles pour l'équation elliptique. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 205–212. (Russian summary, unbound insert)

This paper is concerned with generalizing the results obtained in a previous paper [Ann. Polon. Math. **3** (1957), 247–284; MR **19**, 282]. In the 1957 paper, the non-linear boundary condition was of the form

$$\frac{\partial u}{\partial T_P}+g(P)u(P)=\Phi(P,u(P)),$$

where $\partial/\partial T_P$ denotes the transversal derivative at some point P of the boundary. This condition is now replaced by

$$\frac{\partial u}{\partial T_P}+g(P)u(P)=\Phi(P,u(P),u_{s_P^{(1)}}(P),\cdots,u_{s_P^{(k)}}(P))$$

where $\{s_P^{(1)}\}$, $\{s_P^{(2)}\}$, $\cdots$ are fields of vectors tangential to the boundary surface and where $u_{s_P^{(k)}}(P)$ denotes the limiting value of the derivative in the tangential direction $s_P^{(k)}$ at P.

As the proofs are very similar to those used in a paper dealing with the problem of tangential derivatives for a normal parabolic equation [Ann. Sci. École Norm. Sup. Paris (3) **75** (1958), 19–35; MR **20** #4085], the author presents his new results "d'une façon abrégée".

E. T. Copson (St. Andrews)

5808:

Morrey, Charles B., Jr. On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. II. Analyticity at the boundary. Amer. J. Math. **80** (1958), 219–237.

This paper is devoted to systems of analytic, non-linear, partial differential equations, which are strongly elliptic in the sense of L. Nirenberg [Comm. Pure Appl. Math. **8** (1955), 648–674; MR **17**, 742]. It is shown that any solution of such a system, which possesses a sufficient number of Hölder-continuous derivatives, and which has analytic Dirichlet data along an analytic portion of the boundary of its domain, can be extended analytically across that portion of the boundary. The scheme of proof is like that set forth in part I [Amer. J. Math. **80** (1958), 198–218; MR **21** #5070] to establish analyticity in the interior of the domain. Generalized potentials relating to a certain elliptic system of linear equations with constant coefficients again play an important role; the potentials now, however, having to satisfy boundary conditions in addition to the previous requirements.

A. Douglis (College Park, Md.)

5809:

Krzyżański, M. Certaines inégalités relatives aux solutions de l'équation parabolique linéaire normale. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 131–135. (Russian summary, unbound insert)

Let D be an unbounded domain lying between the hyperplanes $t=0$ and $t=T>0$, and consider the linear parabolic equation

$$(*)\quad \sum_1^m a_{ij}(X,t)U_{x_ix_j}+\sum_1^m b_j(X,t)U_{x_j}$$
$$+c(X,t)U-U_t=f(X,t),$$

where the coefficients satisfy the following conditions: (i) $\sum a_{ij}\lambda_i\lambda_j$ is positive definite for (X,t) in D; (ii) $c\leq\alpha\sum x_j^2+\beta$, (X,t) in D; (iii) $|b_j|\leq\bar\alpha\sum|x_j|+\bar\beta$, $(j=1,2,\cdots,m)$. Let $u(X,t)$ be a solution of (*) which is of class C^2 in D, continuous in $D+\Sigma$ and such that

$$|U(X,t)|\leq M\exp k_0\sum x_j^2\quad\text{for }(X,t)\text{ in }D+\Sigma,$$

where Σ is the hyperplane $t=0$. It is shown that if $c(X,t)\leq 0$ and $f(X,t)\leq 0$ (or $f(X,t)\geq 0$) for (X,t) in D, and if $U(X,t)\geq -M$ $[U(X,t)\leq M]$, $M\geq 0$, on Σ, then $U(X,t)\geq -M$ $[U(X,t)\leq M]$ in D.

The author had obtained this same result in an earlier paper [Ann. Polon. Math. **4** (1957), 93–97; MR **19**, 1179] under more restrictive conditions.

C. G. Maple (Ames, Iowa)

5810:

Phillips, R. S. Dissipative operators and parabolic partial differential equations. Comm. Pure Appl. Math. **12** (1959), 249–276.

The author uses his techniques of dissipative operators [see Trans. Amer. Math. Soc. **86** (1957), 109–173; **90** (1959), 193–254; MR **19**, 863; **21** #3669] to attack the Cauchy problem for second order parabolic equations in several spatial variables. Although he finds it necessary to extend slightly his abstract theory, the principal concepts are to be found in his previous papers. For the problem at hand a boundary space is defined whose elements characterize all the well-set Cauchy problems with energy decreasing solutions. In particular, the boundary space for the case of one spatial variable is described in detail.

G. Hufford (Seattle, Wash.)

5811:

Szarski, J. Sur la limitation et l'unicité des solutions des problèmes de Fourier pour un système non linéaire d'équations paraboliques. Ann. Polon. Math. **6** (1959), 211–216.

In a previous note [same Ann. **2** (1955), 237–249; MR **17**, 626] the system of parabolic partial differential equations

$$\frac{\partial z_i}{\partial t}=f_i\Big(t,x_1,\cdots,x_n,z_1,\cdots,z_m,$$
$$\frac{\partial z_i}{\partial x_1},\cdots,\frac{\partial z_i}{\partial x_n},\cdots,\frac{\partial^2 z_i}{\partial x_j\partial x_k},\cdots\Big)\quad(i=1,2,\cdots,m),$$

where the ith equation does not contain derivatives of $z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_m$, was studied. Limitations on the solutions and conditions for uniqueness of the solution were obtained. Similar theorems are now proved in more general situations. *E. C. Titchmarsh* (Oxford)

5812:

Yosida, Kôsaku. An operator-theoretical integration of the wave equation. J. Math. Soc. Japan 8 (1956), 79–92.

An operator-theoretic treatment is given of the Cauchy problem for the wave equation in m-dimensional euclidean space:

$$(1)\qquad \partial^2 u(t, x)/\partial t^2 = Au(t, x), \quad u(0, x) = f(x), \quad u_t(0, x) = g(x),$$

where A is the elliptic differential operator,

$$(2)\qquad a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b^i(x)\frac{\partial}{\partial x_i} + c(x), \quad x = (x_1, \cdots, x_m).$$

If we set $v = u_t$, this equation is equivalent to the matricial equation

$$(3)\qquad \frac{\partial}{\partial t}\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix},$$

under the initial conditions

$$(4)\qquad \begin{pmatrix} u(0, x) \\ v(0, x) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.$$

An appropriate Banach space is introduced and the theory of semi-groups of linear operators is applied to (3). It is shown that the operator $\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ generates a group $\{T_t\}$, $-\infty < t < \infty$, such that

$$(5)\qquad T_t\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix}$$

provides a solution of (3) and (1), if the initial functions f, g are suitably prescribed. This method of solution proves the solvability of the Cauchy problem (1) without using the classical Cauchy-Kowalewski existence theorem, or the Laplace-Fourier transform theory.

J. Elliott (New York, N.Y.)

5813:

Ladyženskaya, O. A. Solution in the large of the Cauchy problem for non-stationary plane flow of a viscous incompressible fluid. Trudy Moskov. Mat. Obšč. 8 (1959), 71–81. (Russian)

In einer Arbeit von A. A. Kiselev und O. A. Ladyženskaya [Izv. Akad. Nauk SSSR. Ser. Mat. **21** (1957), 655–680; MR **20** #6881] ist die Existenz einer eindeutigen Lösung des Randwertproblems für eine dreidimensionale zähe inkompressible Flüssigkeitsströmung bewiesen worden. Die dabei in Anwendung gebrachte Methode kann auf allgemeinere Differentialgleichungen der Gestalt

$$\frac{\partial \mathbf{u}}{\partial t} - \nu(x, t)\Delta \mathbf{u} + (u_k + q_k(x, t))\frac{\partial \mathbf{u}}{\partial x_k} = \operatorname{grad} p + \mathbf{f}(x, t), \qquad \operatorname{div} \mathbf{u} = 0,$$

ausgedehnt werden. Dabei handelt es sich zunächst um Lösungen lokalen Charakters und kleine Werte der Reynoldschen Zahl. Die Existenz von Lösungen des in diesem Zusammenhang vorliegenden Cauchyschen Problems hat bereits J. Leray untersucht [J. Math. Pures Appl. (9) **12** (1933), 1–82]. In der vorliegenden Arbeit handelt es sich um einen neuen Beweis des Existenzsatzes für die in Rede stehenden Differentialgleichungen. Ist ν der Koeffizient der Zähigkeit der Flüssigkeit, so wird die Strömung durch die Stromfunktion $\psi(x_1, x_2, t)$ und die Bedingungen

$$L\psi \equiv \frac{\partial \Delta\psi}{\partial t} - \nu\Delta^2\psi + \psi_{x_1}\frac{\partial \Delta\psi}{\partial x_2} - \psi_{x_2}\frac{\partial \Delta\psi}{\partial x_1} = f(x, t)$$

$$\psi|_{t=0} = \psi_0(x_1, x_2)$$

charakterisiert. Die Verfasserin trifft die vereinfachenden Annahmen $\nu = \text{const} > 0$, $f \equiv 0$ und unterwirft die Funktion ψ_0 der Bedingung

$$\int_E \Big(\psi_0{}^2 + \sum_i \psi_{0x_i}^2 + \sum_{i,j} \psi_{0x_i x_j}^2\Big)dx < \text{const}.$$

ψ_0 ist meßbar im Sinne von Sobolev. Um den Grenzfall $|x| \to \infty$ zu erfassen, genügt es, das Problem für $|x| \leqq R$ zu behandeln und mit den Ergebnissen den Grenzübergang $|x| \to \infty$ zu vollziehen. Für die Beweise der Existenz- und Eindeutigkeitssätze wird das Zylindergebiet $Q_R \equiv (|x| \leqq R, t \geqq 0)$ zu Grunde gelegt und für die Lösungsfunktion $\psi = \psi^R(x, t)$ die Rand- und Anfangsbedingungen

$$\psi^R|_{|x|=R} = \Delta\psi^R|_{|x|=R} = 0$$

bzw.

$$\psi^R(x, 0) = \psi_0(x)\xi_R(|x|) \equiv \psi_{0R}(x)$$

vorgeschrieben. Dabei ist $\xi(|x|)$ zweimal stetig differenzierbar, für $|x| \leqq R-2$ vom Wert 1, für $|x| \geqq R-1$ vom Wert Null und beliebig nichtnegativ für $x \in [R-2, R-1]$. Das Problem ist eindeutig lösbar im ganzen Zylinderbereich. Der Existenzbeweis wird nach einer von Galerkin angegebenen Methode geführt. Auch das Cauchysche Problem ist für $t \geqq 0$ eindeutig lösbar. Die Lösung $\psi(x, t)$ ist in beliebigen Teilbereichen der Halbebene $t \geqq 0$ quadratisch integrierbar mit Ableitungen nach den x_i bis zur dritten Ordnung. Die ersten und zweiten Ableitungen der Lösung sind in x_1, x_2 überall quadratisch integrierbar mit beliebigen fest gewählten Werten $t \geqq 0$. Die Ableitungen dritter Ordnung sind im Streifen $0 \leqq t \leqq T$ quadratisch integrierbar. Ferner genügt die Funktion ψ bei beliebigen $T \geqq 0$ der Gleichung

$$\int_0^T \int_E [\psi_{x_i}\Phi_{x_i t} - \nu\Delta\psi\Delta\Phi - \psi_{x_1}\Delta\psi\Phi_{x_2} + \psi_{x_2}\Delta\psi\Phi_{x_1}]dx\,dt + \int_E \psi_{0x_i}\Phi_{x_i}(x, 0)dx = 0.$$

Dabei ist Φ eine beliebig quadratisch summierbare Funktion. Ihre Ableitungen sind im Streifen $0 \leqq t \leqq T$ ebenfalls quadratisch summierbar und verschwinden für $t = T$. Ferner ist die Normierungsbedingung $\psi(0, t) = \psi_0(0)$ erfüllt. Die Funktionen $\int_E \Delta\psi(x, t)w(x)dx$ sind stetig in t für entsprechend gewählte Funktionen $w(x)$.

M. Pinl (Cologne)

5814:

Scott, E. J. An eigenfunction series solution of a certain hyperbolic partial differential equation. SIAM Rev. 1 (1959), 160–166.

The problem is to solve

$$(*)\qquad \frac{\partial}{\partial x}\left[A(x)\frac{\partial u}{\partial x}\right]-B(x)u-C(x)\frac{\partial^2 u}{\partial t^2}=0,$$

where $A>0$, $B\geqq 0$, $C>0$, under the initial conditions

$$u=u_0(x),\quad u_t=u_1(x)\quad (t=0,\ a<x<b)$$

and the non-homogeneous linear boundary conditions

$$\alpha_1 u(a,t)+\alpha_2 u(b,t)+\alpha_3 u_x(a,t)+\alpha_4 u_x(b,t)=f(t),$$
$$\beta_1 u(a,t)+\beta_2 u(b,t)+\beta_3 u_x(a,t)+\beta_4 u_x(b,t)=g(t),$$

where the α_i, β_i are constants and not all of the determinants $\alpha_i\beta_j-\alpha_j\beta_i$ are zero.

This is transformed into an equivalent problem with $f(t)$, $g(t)$ identically zero, but with the right-hand-side of (*) replaced by $\sum_{i=1}^{4}F_i(x)G_i(t)$ where F_i, G_i are known functions. A formal solution is then found of the form $\sum_1^\infty X_n(x)T_n(t)$, where the $X_n(x)$ are the eigenfunctions of

$$\frac{d}{dx}\left[A(x)\frac{dX}{dx}\right]-B(x)X+\lambda C(x)X=0.$$

E. T. Copson (St. Andrews)

5815:

Či, Min′-yu. The Cauchy problem for a class of hyperbolic equations with initial data on a line of parabolic degeneracy. Acta Math. Sinica 8 (1958), 521–530. (Chinese. Russian summary)

The author proves the existence and uniqueness of the solution of the equation $y^2u_{xx}-u_{yy}+au_x=0$, $a=$const., with given initial values $u(x,0)=\tau(x)$, $u_y(x,0)=\nu(x)$. Let $\beta=\frac14(1+a)$, $\beta'=\frac14(1-a)$, then it is assumed that $\tau(x)$ is of class $C_{[1-\beta']+1}$ and $\nu(x)$ of class $C_{[\beta]+1}$. The explicit solution is first obtained for $0<a<1$, then extended to the cases of $a\neq 1, 3 \pmod 4$ by defining the otherwise divergent integrals through analytic continuation of the fractional exponents which occur in the explicit formula. By a similar method the solutions for the cases of $a=1$ and $a=3 \pmod 4$ are obtained.

Yu Why Chen (New York, N.Y.)

5816:

Wuang, Kuang-ying. Properties of the solutions of linear partial differential equations with singular coefficients in the neighborhood of singular line (singular surface). Acta Math. Sinica 7 (1957), 590–630. (Chinese. English summary)

In this paper the equation $L(u)=0$ is treated in the x, y plane with the initial line $x-y=0$, where

$$L(u)=u_{xy}+(\beta'/(x-y)+a(x,y))u_x+(\beta/(x-y)+b(x,y))u_y$$
$$+(\gamma/(x-y)^2+d(x,y)/(x-y)+c(x,y))u,$$

with constants β, β' and γ. The general solution u has the form $u=(x-y)^{\rho_1}v_1(x,y)+(x-y)^{\rho_2}v_2(x,y)$ with ρ_1 and ρ_2 satisfying $F(\rho)=\rho^2+\rho(\beta+\beta'-1)-\gamma=0$. v_1 and v_2 are solutions of equations similar to that satisfied by u, except with $\gamma=0$. With given initial values $v_i(x,x)=v_i(x)$, the existence and uniqueness of regular solutions v_i are proved for $\beta+\beta'=0$ (and $\beta+\beta'>0$). The result is applied to the equation

$$L_t u=u_{tt}+(\beta/t+B(t))u_t+(\gamma/t+q(t))u=u_{xx}$$

with $\beta>0$, to obtain a "transmutation" operator of L_t and d^2/dt^2.

Yu Why Chen (New York, N.Y.)

5817:

Gol′dberg, V. N. On nonlinear mixed problems for hyperbolic equations. Dokl. Akad. Nauk SSSR **127** (1959), 949–952. (Russian)

The equation has principal part $u_{tt}-u_{xx}$ and small, smooth nonlinear terms. It is solved in $0\leqq x,\ t\leqq 1$, with Cauchy data on $t=0$ and boundary conditions on $x=0$ and $x=1$, by the integral equation routine.

P. Ungar (New York, N.Y.)

5818:

Kisyński, Jan. Sur l'existence et l'unicité des solutions des problèmes classiques relatifs à l'équation $s=F(x,y,z,p,q)$. Ann. Univ. Mariae Curie-Skłodowska. Sect. A **11** (1957), 73–112 (1959). (Polish and Russian summaries)

This paper is concerned with the partial differential equation $z_{xy}=F(x,y,z,z_x,z_y)$. Here F is continuous for $(x,y)\in R$: $0\leqq x\leqq a$, $0\leqq y\leqq b$, and arbitrary (z,p,q) and satisfies

$$(1)\quad |F(x,y,z,u,v)-F(x,y,Z,U,V)|\leqq w(|z-Z|+|u-U|+|v+V|),$$

and $w(t)>0$ is a non-decreasing function satisfying $\int_{+0}dt/w(t)=\infty$. He proves existence and uniqueness for a problem of the Goursat type and for a problem which generalizes both the Cauchy and Darboux problems. Existence is obtained on the entire rectangle R, although it is not assumed that F is bounded. Condition (1) can be weakened if the uniqueness assertion is omitted. The existence proofs involve a reduction of the problem to a functional equation (essentially for z_{xy}) and an application of a fixed point theorem. Uniqueness follows from a theorem to the effect that if $r(x,y)$ is continuous on R and

$$0\leqq r(x,y)\leqq w\left(\int_0^x r(s,y)ds+\int_0^y r(x,t)dt\right),$$

then $r\equiv 0$.

P. Hartman (Baltimore, Md.)

5819:

Conlan, James. The Cauchy problem and the mixed boundary value problem for a non-linear hyperbolic partial differential equation in two independent variables. Arch. Rational Mech. Anal. **3**, 355–380 (1959).

For a square domain $\bar{D}$ in the xy-plane, the author is concerned with the existence proof of (1) Cauchy's initial value problem [see, for instance, Hartman and Wintner, Amer. J. Math. **74** (1952), 834–864; MR **14**, 475] and (2) the mixed boundary value problem of the equation $u_{xy}=f(x,y;u,u_x,u_y)$ by prescribing respectively the continuous Cauchy data on the diagonal and the continuously differentiable data of u on the diagonal as well as on one of the sides. Here $f(x,y;u,p,q)$ is assumed to be continuous and bounded, and to satisfy the Lipschitz condition with respect to p and q in the product space $\bar{D}\times S$ where S: $(-\infty<u,p,q<\infty)$. The basic idea is to use a variation of Euler-Cauchy polygon method as introduced by J. B. Diaz [same Arch. **1** (1958), 357–390; MR **21** #2803] for the procedure and to apply Arzelà's theorem for the convergence proof. It should be remarked that the condition of boundedness of f in the whole product space (the condition c in the theorems) excludes the

possibility of applying the result directly to the existence of the solution of the same problem of the simplest case that $f \equiv u$. *S. S. Shu* (Lafayette, Ind.)

5820:

Dobrescu-Purice, Lucia. Solutions singulières sur caractéristiques régulières, d'une classe d'équations aux dérivées partielles linéaires du quatrième ordre. Acad. R. P. Romîne. Stud. Cerc. Mat. **10** (1959), 183–218. (Romanian. Russian and French summaries)

$$E(u) = a^{ijkl}\frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l} + \cdots = 0$$

is a linear partial differential equation of order 4 in n variables $x^1, \cdots, x^n$. The author assumes that the characteristic form, $A = a^{ijkl}p_ip_jp_kp_l$, is the product of two quadratic forms, $P = a^{ij}p_ip_j$ and Q, and considers a characteristic surface $G=0$ which belongs to P so that $a^{ij}(\partial G/\partial x^i)(\partial G/\partial x^j)=0$. She considers solutions of the form $u = UG^p$ where U is holomorphic in some neighborhood of $G=0$. The paper contains a detailed investigation of such solutions as well as of solutions involving log G.

A. Erdélyi (Pasadena, Calif.)

5821:

Browder, Felix E. Eigenfunction expansions for non-symmetric partial differential operators. II. Amer. J. Math. **81** (1959), 1–22.

Soit B un opérateur différentiel défini dans un ouvert G, tel que $(Bu, u) > 0$ pour toute u (non nulle) de $C_c^\infty(G)$. Soit H_B l'espace complété de $C_c^\infty(G)$ muni du produit scalaire (Bu, v). Supposons l'injection de H_B dans $\mathscr{D}'(G)$ continue. Soit L un autre opérateur différentiel. On définit alors une réalisation A de (L, B) dans H_B: $(Lu, v) = (BAu, v)$ pour toute $v \in H_B$, où le domaine de définition de A contient $C_c^\infty(G)$ comme sous-espace.

Le travail antérieur [même J. **80** (1958), 365–381; MR **20** #1064] s'améliore en deux points. D'abord, la classe des opérateurs sous-normaux (subnormal) s'étend à celle des opérateurs ayant des réalisations de la forme $T = S + N$, où S est un opérateur scalaire attaché à une mesure spectrale: $[Su, v] = \int \zeta d[E_\zeta u, v]$, N est un opérateur semi-nilpotent (c'est-à-dire que, pour toute u, il existe $j(u)$ tel que $N^{j(u)}u = 0$). L'A. montre que ces opérateurs admettent une décomposition analogue à la forme triangulaire dans le cas des matrices ordinaires. Ensuite, le théorème de Birkhoff-Gelfand est remplacé avantageusement par un théorème des noyaux que voici: Soit V une application continue de $L^p(m)$ ($1 \leq p < \infty$, m étant une mesure finie sur les ensembles boréliens dans C^1) dans $\mathscr{D}'(G)$. Il existe alors une fonction faiblement mesurable $f(\zeta)$, $\zeta \in C^1$, à valeurs dans $\mathscr{D}'(G)$, telle que $(V\alpha, u) = \int (f(\zeta), u)\alpha(\zeta)dm(\zeta)$, pour toute $u \in C_c^\infty(G)$ et pour toute $\alpha(\zeta) \in L^p(m)$. L'A. le montre en utilisant le théorème de Dunford-Pettis. En utilisant ce théorème, il est montré que la classe d'opérateurs susdite admet un développement suivant les fonctions propres. Naturellement, la notion de la fonction propre doit être prise au sens généralisé. Finalement l'A. montre que, dans certains cas favorables (qu'il nomme paire hypo-elliptique), les fonctions propres deviennent fonctions indéfiniment différentiables. *S. Mizohata* (New York, N.Y.)

POTENTIAL THEORY

See also 5711a, 5727, 5746, 5805, 5806, 5992, 5996, 5997, 6005.

5822:

Doob, J. L. A relativized Fatou theorem. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 215–222.

On sait depuis longtemps que dans une boule de R^n ($n \geq 2$) une fonction harmonique >0 admet p.p. à la frontière une limite finie non tangentielle (c.à.d. prise dans un cône axé sur le rayon et de demi-angle $<\pi/2$). Inspiré par les recherches récentes sur l'allure des fonctions surharmoniques >0 à la frontière de Martin, l'A. revient à la question classique en considérant le rapport u/h de deux fonctions harmoniques >0 et les mesures associeés U et H dans la représentation de Poisson-Stieltjes. Il montre l'existence d'une limite finie non tangentielle H-p.p, et elle vaut H-p.p la dériveé de la composante absolument continue (relativement à H) de U par rapport à H. (Un example montre que cela ne s'étend pas à u surharmonique >0, même pour des limites radiales.) L'A. utilise une série de lemmes plus ou moins difficiles {l'A. demande qu'on signale dans l'énoncé du lemme 3.2 une rectification banale consistant à remplacer pour deux points d'une inégalité de type Harnack, la condition qu'ils appartiennent à un segment de cône par celle qu'ils appartiennent à une boule centrée sur l'axe du cône et située dans le segment}. La difficulté que cause le remplacement de $h = \text{const}$ par h quelconque disparaîtrait dans le cas $n=2$, où l'on peut préciser davantage comme dans le cas de $h = \text{const}$.

M. Brelot (Paris)

5823:

Boboc, N.; et Radu, N. Une classe de fonctions définies sur des variétés topologiques triangulables. Mesures associées. Fonctions de Green associées. Rev. Math. Pures Appl. **3** (1958), 309–323.

Cet article constitue une généralisation de la théorie des fonctions harmoniques sur une variété différentiable V de dimension n. Les A. supposent que sur chaque domaine $D \subset V$ on a défini un espace vectoriel $\mathscr{L}(D)$ de fonctions numériques continues, n'admettant aucun maximum local et tel que: (i) le problème de Dirichlet soit résoluble pour un domaine jordanien contenu dans un voisinage de coordonnées; (ii) toute suite croissante et bornée $u_n \in \mathscr{L}(D)$ soit uniformément convergente; (iii) pour chaque sous-variété W^p de dimension $p \leq n-2$, il existe une "solution élémentaire" tendant vers $+\infty$ sur W^p. Le but de l'article est d'établir l'existence de la "mesure harmonique" d'un compact, et de la fonction de Green.

Note du ref. On doit à M. Brelot [C. R. Acad. Sci. Paris **245** (1957), 1688–1690; **246** (1958), 2334–2337, 2709–2712; MR **21** #5094, #5095] une axiomatique de la théorie des fonctions harmoniques, valable sur un espace localement compact connexe, qui ne suppose pas l'existence à priori de fonctions élémentaires. [Voir aussi: R. M. Hervé et M. Brelot, ibid. **247** (1958), 1956–1959; MR **21** #5096.]

J. Lelong (Paris)

5824:

Hunt, G. A. Markoff processes and potentials. III. Illinois J. Math. **2** (1958), 151–213.

Pour certaines définitions, nous renvoyons à l'analyse des deux premières parties de ce travail, qui renouvelle la théorie du potentiel en la plaçant dans un cadre très général et très naturel [même J. **1** (1957), 44–93, 316–369; MR **19**, 951]. En particularisant un peu la situation étudiée antérieurement, l'auteur construit une théorie possédant encore un caractère de très grande généralité, mais dont les énoncés sont plus précis et ont un aspect plus familier.

On se donne une fois pour toutes un semi-groupe fortement continu d'opérateurs linéaires positifs H_τ, de norme ≤ 1, sur $\mathscr{C}(\mathscr{X})$, espace des fonctions réelles continues sur $\mathscr{X}$ localement compact séparable, tendant vers 0 à l'infini. On se donne également une mesure positive ξ excessive ($\xi H_\tau \leq \xi$ quel que soit $\tau > 0$; dans tous les cas classiques tels que potentiels newtoniens, de M. Riesz, de la chaleur, etc., ξ n'est autre que la mesure de Lebesgue), et on suppose que les mesures $H_\tau(r, \cdot)$, ou plus exactement les régularisées $H(\gamma, r, \cdot)$ définies par $H(\gamma, r, B) = \int_0^\infty \gamma(\tau) H_\tau(r, B) d\tau$ sont absolument continues par rapport à ξ, leur densité $h(\gamma, r, s)$ étant dans $\mathscr{C}(\mathscr{X})$ (en tant que fonction de r et en tant que fonction de s). Alors les opérateurs $\hat{H}_\gamma$ définis par $f\hat{H}_\gamma(s) = \int f(r)\xi(dr)h(\gamma, r, s)$ sont eux-mêmes déduits par régularisation d'un autre semi-groupe $\hat{H}_\tau$; on étudie brièvement les processus correspondants et les systèmes de temps terminaux en dualité.

On prend pour "noyau" la fonction U définie par $U(r, s) = h(\gamma, r, s)$ avec $\gamma(\tau) = 1$ quel que soit $\tau > 0$; elle est semi-continue inférieurement par rapport à chacun des points r et s; ce "noyau-fonction" est la densité du "noyau-opérateur" $\int_0^\infty H_\tau d\tau$ considéré antérieurement. Le potentiel droit $U\mu$ d'une mesure μ sur $\mathscr{X}$ est la fonction $U\mu(r) = \int U(r, s)\mu(ds)$; c'est la densité par rapport à ξ du "potentiel-mesure" construit à l'aide du semi-groupe $\hat{H}_\tau$.

Une hypothèse de régularité bien naturelle est la suivante: pour tout compact F les intégrales $\int_F U(r, s)\xi(ds)$ et $\int_F U(r, s)\xi(dr)$ sont bornées en r et en s. Il est remarquable que cela entraîne la plupart des énoncés bien connus de la théorie classique, notamment un théorème de décomposition de F. Riesz, un énoncé précis du balayage ou de l'extrémisation sur un ensemble analytique, et les théorèmes de convergence sur les suites monotones de potentiels (mais, dans le cas des suites décroissantes, on peut seulement dire que la limite est égale à un potentiel presque partout pour ξ). Sous les mêmes hypothèses on définit et on étudie la capacité d'un ensemble relativement à un couple de fonctions excessives droite et gauche données (i.e., H_τ et $\hat{H}_\tau$ excessives); on obtient une fonction d'ensemble alternée d'ordre infini au sens de Choquet [Ann. Inst. Fourier, Grenoble **5** (1953–54), 131–295; MR **18** 295], qui se réduit à la capacité classique lorsque les fonctions excessives données sont toutes deux identiques à 1.

Les hypothèses faites n'entraînent pas la validité du lemme de Kellogg, à savoir l'existence d'au moins un point régulier pour un compact non négligeable quelconque. On donne trois énoncés qui, sous les hypothèses déjà faites, sont équivalents au lemme de Kellogg; parmi eux se trouve le principe de régularité d'Evans-Vasilesco (la continuité de la restriction d'un potentiel borné au support des masses entraîne la continuité du potentiel dans tout l'espace). On donne encore diverses conditions suffisantes pour ce principe; en particulier la symétrie en r et s des fonctions $h(\gamma, r, s)$ suffit; dans ce dernier cas on peut adapter les résultats de H. Cartan relatifs à la théorie newtonienne fine [Bull. Soc. Math. France **73** (1945), 74–106; MR **7**, 620]. Le mémoire se termine par quelques remarques concernant la théorie relative.

J. Deny (Plaiseau)

5825:

Doob, J. L. Discrete potential theory and boundaries. J. Math. Mech. **8** (1959), 433–458; erratum 993.

Ce travail comprend deux parties distinctes. Dans la première l'auteur esquisse une théorie générale du potentiel par rapport à un noyau de la forme $g = \sum p^n$, où p est un opérateur sous-stochastique sur un espace mesurable R. Ces noyaux, que le reviewer appelle élémentaires, sont des cas particuliers de ceux considérés par G. Hunt [Illinois J. Math. **1** (1957), 44–93, 316–369; MR **19**, 951], à savoir $\int_0^\infty H_t dt$, où les H_t constituent un semi-groupe d'opérateurs sous-stochastiques. La théorie du potentiel par rapport à un noyau élémentaire est remarquablement simple, et on peut donner des résultats plus précis que dans le cas général. Par exemple, les fonctions "superregular" u (les analogues des surharmoniques dans le cas newtonien; Hunt les appelle excessives) sont définies par une seule inégalité: $pu \leq u$; si u est ≥ 0, on a un théorème de décomposition de Riesz: $u = gv + h$, où h est régulière ≥ 0 ($h = ph$) et gv est le potentiel engendré par la fonction $v \geq 0$ (une décomposition analogue n'est pas vraie pour tout noyau de Hunt; contre-exemple: le noyau newtonien; par contre la décomposition de Riesz, donnée dans le texte, des mesures "superregular" μ ($\mu p \leq \mu$) s'étend au cas général). L'auteur donne un théorème du balayage pour les potentiels de fonctions et pour les potentiels de mesures. Une fonction "superregular" qui joue un rôle important est celle qui vaut en $x \in R$ la probabilité pour qu'un processus (discret) partant de x rencontre un ensemble donné A (généralisation naturelle de la notion de potentiel d'équilibre de A dans le cas classique). A signaler encore une définition simple des notions de régularité et de "superregularity" sur une partie mesurable de R.

La seconde partie est consacrée aux problèmes aux limites, problèmes non abordés dans le travail cité de Hunt. Pour cela on se borne au cas où R est discret et dénombrable. On construit deux sortes de frontières idéales ("the Martin exit boundary" et "the Martin entrance boundary") qui sont moins vastes que celles introduites, dans une situation analogue, par W. Feller [Trans. Amer. Math. Soc. **83** (1956), 19–54; MR **19**, 892]; les définitions de l'auteur, directement inspirées de celles données dans le cas newtonien par R. S. Martin [ibid. **49** (1941), 137–172; MR **2**, 292], se prêtent très bien à la représentation intégrale à la Martin des fonctions "superregular" positives, à la résolution du problème de Dirichlet, et à la généralisation de divers résultats récents de théorie newtonienne fine [M. Brelot, J. Math. Pures Appl. (9) **35** (1956), 297–335; MR **20** #6607; J. L. Doob, Bull. Soc. Math. France **85** (1957), 431–458; L. Naïm, Ann. Inst. Fourier, Grenoble **7** (1957), 183–281; MR **20** #6608].

J. Deny (Paris)

5826:

Nečas, Jindřich. Solution du problème biharmonique pour le coin infini pas convexe. Časopis Pěst. Mat. **84** (1959), 90–98. (Czech. Russian and French summaries)

In an earlier paper [same Časopis **83** (1958), 257–286, 399–424; MR **21** #1462] the author employed the theory

of the Mellin transform to prove the existence and uniqueness of a function biharmonic in the interior of an infinite convex wedge, with the function and its normal derivative assuming, in a specially prescribed sense, specific boundary values. The same theory is now extended to the case of an infinite non-convex wedge.

J. F. Heyda (Cincinnati, Ohio)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 5744, 5763, 5852.

5827:

★**Levy, H.; and Lessman, F. Finite difference equations.** Sir Isaac Pitman & Sons, Ltd., London, 1958. vii+278 pp. 37s. 6d.

Difference equations arise naturally as the description for many problems in the theory of probability, in queueing theory, in the study of networks, in statistics, and in many other fields where sequential relations occur at discrete values of the variables. In this book the authors are concerned with methods for solving difference equations and, except for touching upon the subject, are not concerned with approximating solutions to problems involving continuous variables.

The first three chapters contain some of the basic knowledge of elementary difference operators, interpolation and extrapolation, and many of the definitions concerned with difference equations. The last five chapters, except for chapter seven, are devoted to methods of solving difference equations (linear difference equations with constant coefficients, general first order difference equations, linear difference equations with variable coefficients, and difference equations associated with functions of two variables). Chapter seven is devoted to specific examples of difference equations.

The stress in the book is on exhibiting methods of solution, and the justification for these methods is often heuristic. For a deeper mathematical appreciation of difference equations and to broaden one's knowledge of this subject one might also read L. Collatz, *Numerische Behandlung von Differentialgleichungen* [Springer, Berlin, 1955; MR **16**, 962]; F. B. Hildebrand, *Introduction to numerical analysis* [McGraw-Hill, New York, 1956; MR **17**, 788]; A. S. Householder, *Principles of numerical analysis* [McGraw-Hill, New York, 1953; MR **15**, 470]; E. Pinney, *Ordinary difference-differential equations* [Univ. of Calif. Press, Berkeley, 1958; MR **20** #4065]; and R. D. Richtmyer, *Difference methods for initial-value problems* [Interscience, New York, 1957; MR **20** #438].

G. W. Evans, II (Menlo Park, Calif.)

5828:

Gołąb, S.; et Schinzel, A. Sur l'équation fonctionnelle

$$f[x+y\cdot f(x)] = f(x)\cdot f(y).$$

Publ. Math. Debrecen **6** (1959), 113–125.

The differentiable real (non-zero) solutions are of the form $f(x)=1+mx$, where m is any real number.

The only continuous solutions not already mentioned are of the form: $f(x)=0$ when $x\leq x_2$, $f(x)=1-x/x_2$ when $x\geq x_2$ $(x_2<0)$; and $f(x)=1-x/x_1$ when $x\leq x_1$, $f(x)=0$ when $x\geq x_1$ $(x_1>0)$; where x_1 and x_2 are any real numbers subject to the stated restrictions.

The only non-microperiodic (i.e., with no arbitrarily small periods) non-trivial (i.e., the set of values of f contains numbers other than -1, 0, $+1$) solutions f are of the form: $f(x)=1+mx$ if $(1+mx)\in G$, $f(x)=0$ if $(1+mx)\notin G$, where G is a multiplicative subgroup (not just $\{-1,+1\}$) of the reals, and m is a non-zero real number.

Every solution not of the previous forms is discontinuous on a half-line. There are non-trivial (non-continuous) microperiodic solutions. No classification of the measurable solutions is yet available. *A. Nijenhuis* (Seattle, Wash.)

5829:

Wagner, R. Eindeutige Lösungen der Funktionalgleichung

$$f[x+f(x)] = f(x).$$

Elem. Math. **14** (1959), 73–78.

Verf. schliesst sich der Aufgabe Nr. 173 [dieselben Elem. **8** (1953), 20] über die im Titel figurierende, auf Euler zurückgehende Funktionalgleichung und ihrer von W. Lüssy [ibid. **9** (1954), 40] gegebenen Lösung an, indem er, im Gegensatz zu W. Lüssy, eindeutige Lösungen sucht.

Sein Vorgehen, das auf einer gewissen Erweiterung schon vorhandener, einen gewissen Wert nicht annehmender Lösungen zu solchen, die auch diesen Wert annehmen, beruht, hat, wie es der Verf. selbst bemerkt, wegen Verwendung des Wohlordnungssatzes keinen konstruktiven Charakter. Deshalb untersucht Verf. auch Lösungen mit abzählbarem Wertevorrat, wo das nicht der Fall ist. Er zeigt u.a., dass eine eindeutige Lösung immer dieser Art ist, falls sie in einem offenen Intervall stetig und positiv und überall nicht-negativ ist (und zwar sind dann die Funktionenwerte gewisse ganzzahlige Vielfache einer positiven Zahl). *J. Aczél* (Debrecen)

5830:

Kuczma, M. On the functional equation

$$\varphi(x)+\varphi[f(x)] = F(x).$$

Ann. Polon. Math. **6** (1959), 281–287.

The author considers the functional equation in the title [cf. R. Raclis, Bull. Math. Sci. Soc. Roumain. Sci. **30** (1927), 101–105, for meromorphic complex solutions], special cases of which were examined by G. H. Hardy [*Divergent series*, Clarendon, Oxford, 1949; MR **11**, 25; p. 77] and H. Steinhaus [Prace Mat. **1** (1955), 276–284; MR **17**, 497]; more general equations were considered under different conditions by N. Gercevanoff [Dokl. Akad. Nauk SSSR **39** (1943), 207–209 {volume and year are erroneously quoted in the present paper}; MR **5**, 185], T. Kitamura [Tôhoku Math. J. **49** (1943), 305–307; MR **8**, 517; cf. a subsequent paper of the present author] and M. Ghermanescu [C. R. Acad. Sci. Paris **243** (1956), 1593–1595; MR **18**, 581]. Let a, b be two consecutive roots of the equation $f(x)=x$. The author considers solutions of the functional equation in such an interval $\langle a,b\rangle$ and proves that there exists an infinite number of solutions continuous in the open interval (a,b) if $f(x)$ and $F(x)$ are continuous and $f(x)$ also strictly increasing in $[a,b]$, while at most one solution is continuous in a too. The author

gives under some further assumptions a series-construction for the several continuous solutions in (a, b) and two sufficient conditions for the existence of a continuous solution in the interval closed on one side.

J. Aczél (Debrecen)

5831:

Guinand, André-Paul. Les équations fonctionnelles de la loi d'associativité ternaire. C. R. Acad. Sci. Paris **249** (1959), 23–24.

Starting with results on the functional equation $f[x, f(y, z)] = f[f(x, y), z]$ for real variables x, y, z [see, e.g., N. H. Abel, J. Reine Angew. Math. **1** (1826), 11–15; *Œuvres complètes*, vol. I. Christiania, 1881, pp. 61–65; J. Aczél, Bull. Soc. Math. France **76** (1949), 59–64; Acta Univ. Szeged Sect. Sci. Math. **13** (1949), 136–139; (1950), 179–189; MR **10**, 685; **11**, 511; **13**, 246; M. Hosszú, Publ. Math. Debrecen **3** (1954), 205–214; **4** (1956), 459–464; MR **17**, 236; **18**, 48; Kuwagaki, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **27** (1953), 225–234; MR **15**, 324], the author considers the functional equations

$$(1)\qquad f[f(x, y, z), u, v] = f[x, y, f(z, u, v)]$$

$$(2)\qquad f[f(x, y, z), u, v] = f[x, f(y, z, u), v] = f[x, y, f(z, u, v)].$$

He gives without proof the following enumeration of the solutions of: (1) $f(x, y, z) = \text{constant}$, (2) $f(x, y, z) = x$, $f(x, y, z) = z$, (3) $f(x, y, z) = \varphi^{-1}[\varphi(x)\varphi(z)]$ and (4) $f(x, y, z) = \varphi^{-1}[\varphi(x)\psi(y)\varphi(z)]$, where φ and ψ are arbitrary univalent functions, φ having also a univalent inverse φ^{-1}. He asserts also that these are all possible solutions of (2) except that (3) does not satisfy (2) and (4) is taken with $\psi(y) = \varphi(y)$ or $\psi(y) = 1/\varphi(y)$ or $\psi(y) = -1/\varphi(y)$. It is not clear what are the suppositions in these theorems, but the author's remark that he uses methods similar to those in the papers of M. Hosszú and A. Kuwagaki quoted above, which, by the way, concern functions of complex variables, seems to show that differentiability conditions are assumed. In any case, if x, y, z, u, v lie in one interval and continuity is assumed, then the solution (4) with $\psi(y) = -1/\varphi(y)$ falls out. On the other hand if this solution is taken into consideration, one does not see why $\psi(y) = -\varphi(y)$ is not considered too.—It might be remarked that (2) is the functional equation of ternary polyadic groups, while (1) figures in the definition of flocks [see, e.g., H. Prüfer, Math. Z. **20** (1924), 166–187; W. Dörnte, ibid. **29** (1928), 1–19; R. Baer, J. Reine Angew. Math. **100** (1929), 199–207; L. Post, Trans. Amer. Math. Soc. **48** (1940), 208–350; MR **2**, 128; J. Certaine, Bull. Amer. Math. Soc. **49** (1943), 869–877; MR **5**, 227; H. Tvermoes, Math. Scand. **1** (1953), 18–30; MR **15**, 98; R. H. Bruck, *A survey of binary systems*, Springer, Berlin-Göttingen-Heidelberg, 1958; MR **20** #76; pp. 36–41]. A detailed exposition of the present results will be awaited with interest.

J. Aczél (Debrecen)

SEQUENCES, SERIES, SUMMABILITY

See also 5702, 5720.

5832:

Wuyts-Torfs, M. On a generalization of the Euler limit method. Simon Stevin **33** (1959), 27–33. (Dutch)

The author considers the inclusion relation

$$(1)\qquad E(r) \subset E(r_n).$$

Here $E(r)$ is the Euler-Knopp transformation and $E(r_n)$ was defined by R. P. Agnew [Amer. J. Math. **66** (1944), 313–338; MR **6**, 46] by

$$\sigma_n = \sum_{k=0}^{n} \binom{n}{k} r_n^k (1-r_n)^{n-k} i_k.$$

Agnew considered the case $0 < r \leqq 1$ and r_n real. Here (1) holds if and only if $r_n \to 0$ and $nr_n \to \infty$. The author considers the complex case. If

$$|r_n| + |r - r_n| = |r|(1+\omega_n), \quad |r - r_n| = |r|(1-\theta_n) \quad (r \neq 0),$$

then (1) holds if and only if $n\omega_n$ is bounded and $\lim n\theta_n = +\infty$ (equivalently $|r - r_n| < |r|$, $\lim n|r_n| = +\infty$). Related conditions are analyzed.

E. Hille (New Haven, Conn.)

5833:

Varshney, O. P. On the relation between harmonic summability and summability by Riesz means of certain type. Tôhoku Math. J. (2) **11** (1959), 20–24.

Harmonic summability of a sequence $\{s_n\}$ is defined by the mapping $s \to y$, where

$$y_n = (b, s)_n/(b, 1)_n = \left(\sum_0^n b_{n-k} s_k\right) \Big/ \left(\sum_0^n b_k\right), \quad b_n = 1/(n+1).$$

It is known that harmonic summability implies summability (C, k) for all $k > 0$, and that the condition $\Delta s_n = O(n^{-\alpha})$ for any $0 < \alpha < 1$ is a Tauberian condition for harmonic summability. In the present paper, it is shown that harmonic summability implies summability by simple typical means with weights $\{p_n\}$, $t_n = (\sum p_k s_k)/P_n$, where $P_n = \sum_0^n p_k = \exp(n^\alpha)$, $0 < \alpha < 1$.

R. C. Buck (Princeton, N.J.)

5834:

Kuttner, B. On a certain quasi-Hausdorff transformation. J. London Math. Soc. **34** (1959), 401–405.

Let $\mu_n = (\lambda n + 1)/(n+1)$. Then the Hausdorff transformation (H, μ_n) is the Mercerian transformation, known to be equivalent to convergence if and only if $\Re\lambda > 0$. However, in the case of the quasi-Hausdorff transformation (H^*, μ_{n+1}), which is a transformation of the form

$$t_n = \lambda s_n + (n+1)(1-\lambda) \sum_{m=n}^{\infty} \frac{s_m}{(m+1)(m+2)},$$

the results which hold are strikingly different from the ones suggested by the close relationship between the Hausdorff transformation (H, μ_n) and the quasi-Hausdorff transformation (H^*, μ_{n+1}) and the author proves that the above transformation is equivalent to convergence if and only if either (a) $\Re\lambda < 0$ or (b) $\lambda \neq 0$ and $\Re(1/\lambda) \geqq 1$. Also, the transformation is shown to be translative for all λ.

M. S. Ramanujan (Ann Arbor, Mich.)

5835:

Yano, Kenji. Notes on Tauberian theorems for Riemann summability. II. Proc. Japan Acad. **35** (1959), 7–12.

In Fortsetzung früherer Untersuchungen [Tôhoku Math. J. (2) **10** (1958), 19–31; MR **20** #1870] beweist der

Verf. mit den Bezeichnungen des dortigen Referats: Aus

$$\sum_{\nu=1}^{n} |S_\nu^\beta| = o(n^{\gamma+1}), \quad \sum_{\nu=n}^{2n} (|S_\nu^b| - S_\nu^b) = O(n^{b+\delta+1})$$

folgt (R, p)-$\sum a_n = 0$, sofern $b \geqq -1$, $b < p-1 < \gamma < \beta$ und $\delta = (p-1-b)(\beta-\gamma)/(\beta-p+1)$ gesetzt ist. Ein ähnlicher Satz wird für Riemann-Cesàro Summierbarkeit bewiesen.

D. Gaier (Giessen)

5836:

Dikshit, G. D. On the absolute summability factors of infinite series. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 191–200.

The reviewer proved the following lemma: If $\sum_1^n |\Delta\sigma_k^\alpha| = O(\log n)$ where σ_k^α is the kth Cesàro mean of order α for $\sum_1^\infty a_n$, then $\sum_1^\infty a_n\{\log(n+1)\}^{-1-\varepsilon}$, $\varepsilon > 0$, is summable $|C, \alpha|$ [Kōdai Math. Sem. Rep. **1954**, 59–62; MR **16**, 464]. The author generalizes this lemma taking $\sum_1^n |\Delta\sigma_k^\alpha| = O(\mu_n)$ instead of $O(\log n)$. Some applications are given.

G. Sunouchi (Evanston, Ill.)

5837:

Tolba, S. E. On the efficiency of T-matrices for bounded sequences and the Cooke-Barnett condition. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 265–274.

The following notation is used: $\mathfrak{S}_N$ is the set of all bounded sequences of real or complex numbers, each sequence of which has N and only N distinct limit points; $\mathfrak{S}_N^*$ denotes the aggregate of the sets $\mathfrak{S}_2, \mathfrak{S}_3, \cdots, \mathfrak{S}_N$; $\mathfrak{S}$ is the set of all bounded divergent sequences of real or complex numbers with any finite number of distinct limit points; $\mathfrak{S} - \mathfrak{S}_N$ is the set of all sequences of $\mathfrak{S}$ after removing those of $\mathfrak{S}_N$; $\mathfrak{S}_N(A)$, $\mathfrak{S}_N^*(A)$, $\mathfrak{S}(A)$ denote the subsets of $\mathfrak{S}_N$, $\mathfrak{S}_N^*$, $\mathfrak{S}$, respectively, which are in the convergence field of the matrix A. By the "Cooke-Barnett condition" the author means the condition given in the following theorem [R. G. Cooke and A. M. Barnett, J. London Math. Soc. **23** (1948), 211–221; MR **10**, 447]: Let $\{z_n\}$ be a bounded sequence with a finite number N of distinct limit points; a sufficient condition that $\{z_n\}$ should be summable by a T-matrix A is that A should be efficient for M particular sequences of 0's and 1's suitably constructed from the sequence $\{z_n\}$, where $M = N$ if $\{z_n\}$ is real, and $M = 2N$ if $\{z_n\}$ is complex. The following results are proved. [2. I] Given a fixed number σ and a sequence $\{z_k\}$ of $\mathfrak{S}_N$, where N is a given integer > 2, there exists a T-matrix which sums $\{z_k\}$ to σ, but which is inefficient for all sequences of $\mathfrak{S}_{N-1}^*$. [2. II] (Extension of the Cooke-Barnett theorem) Let $\{z_k\}$ be a sequence of $\mathfrak{S}_N$; a sufficient condition that $\{z_k\}$ should be summable by a T-matrix A is that A be efficient for $N-1$ particular sequences of 0's and 1's suitably constructed from $\{z_k\}$. If $N = 2$, the condition is necessary; if $N > 2$, the condition is not necessary. Corollary: Let $\{z_k\}$ be a sequence of $\mathfrak{S}_3$ having its distinct limit points represented in the complex plane by non-collinear points. A necessary and sufficient condition that $\{z_k\}$ should be summable by a real T-matrix A is that A should be efficient for two particular sequences of 0's and 1's suitably constructed from $\{z_k\}$. [3. I] Given an integer $N > 2$ and a complex constant σ, there exists a T-matrix A which is inefficient for all sequences of $\mathfrak{S}_{N-1}^*$, and which has the property that if M is any integer $> N$, then at least one sequence of $\mathfrak{S}_M$ is summable-A to σ. [3. II] Given an integer $N > 3$, there exists a T-matrix which is inefficient for all sequences of $\mathfrak{S}_{N-1}^*$ and for all sequences of $\mathfrak{S}_{N+2}, \mathfrak{S}_{N+3}, \cdots, \mathfrak{S}_{2N-2}$, but in the convergence field of which there is at least one sequence of each of $\mathfrak{S}_N$, $\mathfrak{S}_{N+1}$, and $\mathfrak{S}_{2N-1}$. [3. III] If a T-matrix is efficient for one sequence of $\mathfrak{S}_N$, where N is a given integer > 2, then A is also efficient for an infinite number of sequences of $\mathfrak{S}_N$. It may be mentioned that R. Henstock has dealt with the extension of the "Cooke-Barnett" theorem to the case where $\{z_n\}$ has an enumerable, or a continuum, infinity of limit points [R. Henstock, ibid. **25** (1950), 27–33; MR **11**, 429; see also the reviewer's *Infinite matrices and sequence spaces*, Macmillan, London, 1950; MR **12**, 694; pp. 201–204].

R. G. Cooke (London)

5838:

Cowling, V. F.; and Royster, W. C. On the Euler and Taylor summation of Dirichlet and Taylor series. Rend. Circ. Mat. Palermo (2) **7** (1958), 270–284.

A series $\sum_{k=1}^\infty c_k$ is said to be summable by the series-to-series transformation by the matrix $A = (a_{n,k})$ to sum S if $\sum_{n=1}^\infty \sum_{k=1}^\infty a_{n,k} c_k$ converges to S. The transformation is said to be regular if the above double sum is equal to $\sum_{k=1}^\infty c_k$ when $\sum c_k$ converges. To determine the effectiveness of A, two chief approaches have been employed. First, if $\sum_{k=1}^\infty c_k(z)$ represents an analytic function, the effectiveness of A is measured in terms of the domain of regularity D of the function defined by $\sum c_k(z)$; i.e., we determine that subdomain of D in which A sums $\sum c_k(z)$. Frequently the transformed series provides the analytic continuation of $\sum c_k(z)$ to certain parts of D, although this is not necessarily so in all cases. The second approach is to assume that A is effective at some point, say $z = z_0$, i.e., $\sum_{n=1}^\infty \sum_{k=1}^\infty a_{n,k} c_k(z_0)$ converges, and then, in terms of this assumption, attempt to determine other sets in which A is effective. In this paper, the question of the effectiveness of A is approached in the following manner. Let $b_n = b_n(z_0) = \sum_{k=1}^\infty a_{n,k} c_k(z_0)$, and write $g(u) = \sum b_n u^n$; then, instead of assuming that $\sum c_k(z)$ is summable-A at $z = z_0$, i.e., that $\sum b_n(z_0)$ converges, it is assumed that $g(u)$ has certain function-theoretic properties; e.g., that $g(u)$ is regular in a neighbourhood of $u = 0$ and of $u = 1$, or that $g(u)$ is regular in $|u| < 1$ and continuous in $|u| \leqq 1$. These conditions do not imply that $\sum c_k(z)$ is summable-A at $z = z_0$, nor does the A-summability of $\sum c_k(z)$ at $z = z_0$ imply that $g(u)$ is regular at $u = 1$. The authors are concerned with two one-parameter families of series-to-series methods: (i) the Euler-Knopp matrix $a_{n,k} = \binom{n}{k} r^{k+1}(1-r)^{n-k}$ $(0 \leqq k \leqq n)$, $a_{n,k} = 0$ $(k > n)$, which is regular if, and only if, $0 < r \leqq 1$, and (ii) the so-called Taylor matrix $a_{n,k} = 0$ $(0 \leqq k < n)$, $a_{n,k} = (1-\alpha)^n \binom{k}{n} \alpha^{k-n}$ $(k \geqq n)$, which is regular if, and only if, $0 \leqq \alpha < 1$. The results obtained in the paper, which are too lengthy and complicated to state in a review, apply to ordinary Dirichlet series and Taylor series.

R. G. Cooke (London)

5839:

Zeller, Karl. Saturation bei äquivalenten Summierungsverfahren. Math. Z. **71** (1959), 109–112.

Soit $A = (a_{nm})$ une matrice définissant un procédé de sommation de la série $\sum u_m$, $A \cdot \sum u_m = \lim_{n\to\infty} \sum_{m=0}^\infty a_{nm} u_m$, et telle que $\lim_{n\to\infty} a_{nm} = 1$ $(m = 0, 1, \cdots)$, la classe de

saturation dans les espaces B de suites $\xi=\{x_k\}$ ($k=0, 1, \cdots$), où les x_k appartiennent à un espace B et sont des fonctions linéaires et continues de ξ, est l'ensemble des suites

$$\left\| \left\{ \frac{a_{nm}-1}{\rho_n} x_m \right\}_{m=0,1\cdots} \right\| = O(1),$$

lorsque $0<\rho_n=O(a_{nm}-1)$, $(a_{nm}-1)=O(\rho_n)$ ($n\to\infty$; $m=0, 1, \cdots$). Pour toute matrice A telle que $\lim_{n\to\infty} a_{nm}=1$, il existe une matrice $B=(b_{nm})$ définissant un procédé de sommation équivalent et telle que $b_{nm}=1$, pour $n>n(m)$ ($n=0, 1, \cdots$). *J. Favard* (Paris)

5840:

Ostrowski, A. M. Three theorems on products of power series. Compositio Math. **14**, 41–49 (1959).

Given positive sequences $\{S_n\}$ and $\{T_n\}$, and a sequence $\{x_n\}$, form the sequence $xS=\{x_nS_n\}$ and the sequence y, with

$$y_n = (xS, T)_n/(S, T)_n,$$

where the Silverman symbol $(a, b)_n=\sum a_{n-k}b_k$. Then, assuming that $\sum S_n=\infty$ and $\limsup T_{n+1}/T_n\leqq 1$, it is shown that the transformation $x\to y$ is regular and, specifically, that

$$\liminf x_n \leqq \liminf y_n \leqq \limsup y_n \leqq \limsup x_n.$$

The other theorems are related to this one.
R. C. Buck (Princeton, N.J.)

5841:

Chen, Yung-ming. Integrability of power series. Arch. Math. **10** (1959), 288–291.

Suppose that $F(x)=\sum_0^\infty c_nx^n$ ($0\leqq x<1$, $c_n\geqq 0$). Let $S_n=\sum_0^n c_\nu$, and suppose that $S_{n\nu}/S_n\leqq A\Phi(\nu)$ ($n\geqq N$, $\nu=1, 2, 3, \cdots$), where A is an absolute constant and $\Phi(\nu)$ is a positive function satisfying $\sum \Phi(\nu)e^{-\nu}<\infty$. Suppose also that $\eta(x)\geqq 0$, $\eta(x)\in L(0, 1)$, and let

$$a_n = \int_{1-1/n}^1 \eta(x)dx.$$

Then the author proves that $\eta(x)F(x)\in L(0, 1)$ if and only if $\sum c_na_n<\infty$. This theorem is related to results of Boas and González-Fernández [J. London Math. Soc. **32** (1957), 48–53; MR **18**, 896] and the reviewer [ibid. **32** (1957), 58–62; MR **18**, 896], where, however, $\eta(x)$ was assumed increasing and no restriction was placed on S_n. The author also proves other results of a similar nature.
P. B. Kennedy (Cork)

APPROXIMATIONS AND EXPANSIONS

See also 5621, 5702, 5720.

5842:

Du Plessis, N. Concerning the validity of finite difference operations. J. London Math. Soc. **34** (1959), 208–214.

Der Verfasser zeigt einen Weg zur (exakten) Herleitung bekannter Interpolations- und Quadraturformeln. Es sei $\{u_n\}$ eine Zahlenfolge mit $|\Delta^r u_0|<Kh^r$ ($0<h<1$, $\Delta u_0=u_1-u_0$) und $A(h)$ die Menge der Funktionen $\phi(t)=\sum_{r=0}^\infty a_rt^r$ mit ($\|\phi\|=$) $\sum_{r=0}^\infty |a_r|h^r<\infty$. Auf $A(h)$ wird durch $f(t^r)=\Delta^r u_0$, $f(\sum^\infty \lambda_rt^r)=\lim_{n\to\infty} f(\sum^n \lambda_rt^r)$ ein stetiges lineares Funktional f definiert. Es ergibt sich dafür z.B. $\Delta^n u_r=f[(1+t)^rt^n]$. Bestimmten auf die Funktion $u(z)=f[(1+t)^z]$ angewendeten Operatoren entsprechen ($\sim$) bestimmte Funktionen von t; z.B. $\Delta\sim t$, $E\sim(1+t)$, $d/dz=D\sim\log(1+t)$. Indem man Umformungen (etwa Reihenentwicklungen) nicht mit den Operatoren, sondern den zugehörigen Funktionen von t vornimmt, erhält man z.B. die Newtonsche Interpolationsformel, die Trapezregel und die Simpsonsche Regel.—Entsprechende Untersuchungen für andere einfache Formeln.
J. Schröder (Hamburg)

5843:

Malliavin, Paul. L'approximation polynomiale pondérée sur un espace localement compact. Amer. J. Math. **81** (1959), 605–612.

Let E be a locally compact space, $C(E)$ the algebra of complex-valued continuous functions $f(x)$ on E, $C_0(E)$ the subalgebra of functions in $C(E)$ which tend to 0 as x tends to ∞. The norm of $f\in C_0(E)$ is $\|f\|=\sup|f(x)|$ ($x\in E$). Let F be a subset of $C(E)$ separating points of E, $A(F)$ the algebra generated by F and the unit function $I(x)=1$. A non-vanishing function $p(x)$ is called a weight for F, if $pA(F)\subset C_0(E)$ and $pA(F)$ is dense in $C_0(E)$. Write $R(D, M(n))$ for the statement: Let μ be a measure in the complex z-plane with support in the closure of D. If $\int z^nd\mu=0$ and $\int |z^n|\,|d\mu|\leqq M(n)$ ($n=0, 1, 2, \cdots$), then $\mu=0$. Write $S(D, M(n))$ for the statement: $R(D, (M(2^sn))^{2^{-s}})$ holds for $s=0, 1, 2, \cdots$. The main result is theorem 1: If $S(f(E), \|pf^n\|)$ is true for every $f\in F$, then p is a weight. A consequence is Theorem 2: If, for every $f\in F$, $f(E)$ is of plane measure 0, the complement of the closure of $f(E)$ consisting of k domains each of which contains a sector of opening $\geqq\phi$, and

$$\sum \|p(x)f^n(x)\|^{-\pi/\phi n} = \infty,$$

then $p(x)$ is a weight.

The author shows first that the proof can be reduced to the study of approximation by polynomials in n complex variables z_j ($=f_j(x)$) ($j=1, 2, \cdots, n$; $f_j\in C_0(E)$) and reduces this study to the case $n=1$.

{An error (without consequence) is on p. 606 (5th paragraph): if $p=O(p_1)$ and $p(x)$ is a weight, $p_1(x)$ need not be a weight; the converse is true, however: $p(x)$ is a weight if $p_1(x)$ is a weight.} *W. H. J. Fuchs* (Ithaca, N.Y.)

5844:

Weiss, Mary. On the law of the iterated logarithm for uniformly bounded orthonormal systems. Trans. Amer. Math. Soc. **92** (1959), 531–553.

The author considers an orthonormal system $\{\phi_n(x)\}$ of real-valued functions on [0, 1] such that $|\phi_n(x)|\leqq B$, a finite constant, for $0\leqq x\leqq 1$ and $n=1, 2, \cdots$, and she establishes for it the following analogue of the law of the iterated logarithm for independent random variables: there exists a subsequence $\{\phi_{n_k}(x)\}$ and a real-valued function $f(x)$, with $\int_0^1 \{f(x)\}^2dx=1$ and $0\leqq f(x)\leqq B$ on [0, 1], such that for any sequence $\{a_k\}$ of real numbers satisfying

$$A_N = (a_1^2+a_2^2+\cdots+a_N^2)^{1/2}\to\infty \quad (N\to\infty),$$

$$M_N = \max_{k\leqq N} |a_k| = o\{A_N(\log\log A_N)^{-1/2}\},$$

we have (almost everywhere in [0, 1])

$$(*) \qquad \limsup S_N(x)\,(2A_N^2\log\log A_N)^{-1/2} = f(x),$$

where $S_N=\sum_{k=1}^N a_k\phi_{n_k}(x)$. In the case of random variables $f(x)$ was 1 [see A. Kolmogoroff, Math. Ann. **101** (1929), 126–135]. The author's theorem generalizes a result obtained for trigonometric series by R. Salem and A. Zygmund [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 333–338; MR **9**, 181], and it may be compared with the analogue found by G. W. Morgenthaler [Trans. Amer. Math. Soc. **79** (1955), 281–311; MR **17**, 49] for the central limit theorem: in this, too, it was necessary to introduce a function $f(x)$ like that above.

H. P. Mulholland (Exeter)

5845:

Rosenbloom, P. C.; and Widder, D. V. Expansions in terms of heat polynomials and associated functions. Trans. Amer. Math. Soc. **92** (1959), 220–266.

Let $V_n(x, t)$ be defined by

$$\exp(zx+z^2t) = \sum V_n(x, t)z^n/n!,$$

and $W_n(x, t)$ by

$$k(x-2z, t) = \sum W_n(x, t)z^n/n!,$$

where $k(x, t)$ is the fundamental solution of the heat equation. It is shown that V_n and W_n form a biorthogonal system and the questions involved in the expansion of functions in terms of a V_n or W_n series are thoroughly investigated. These questions include the determination of coefficients, the region of convergence, and the characterization of those functions for which the expansions are possible. Asymptotic values of V_n and W_n are investigated and results in the L^2 theory are included.

J. Blackman (Syracuse, N.Y.)

5846:

Suetin, P. K. Polynomials orthogonal over an area. Dokl. Akad. Nauk SSSR **126** (1959), 943–945. (Russian)

Let $\gamma(z)$ be continuous and different from zero interior to a simply-connected region G with boundary Γ, let $n(z)=|\gamma(z)|^2$, and denote by $\{K_n(z)\}$ a set of polynomials such that

$$\iint_G K_n(z)\overline{K_m(z)}n(z)ds = \begin{cases}1, & n = m,\\ 0, & n \neq m.\end{cases}$$

The curve Γ is said to be smooth of order p if for every function $\varphi(z)$ mapping G onto $|z|<1$, $\varphi^{(p)}(z)$ is continuous in a closed region $\bar{G}\subset G$. If $\gamma^{(p)}(z)$ satisfies a Lipschitz condition of order α in $\bar{G}$, and Γ is smooth of order $(p+2)$, then for every closed set $F\subset G$ there exists a constant $C(F)$ such that for each n and $z\in F$, $|K_n(z)|\leq C(F)/n^{p+\alpha}$. If $\gamma(z)$ is analytic in G, then $K_n(z)\leq C_1(F)q^n$, $0<q<1$.

M. Tomić (Belgrade)

5847:

De Bruijn, N. G. Pairs of slowly oscillating functions occurring in asymptotic problems concerning the Laplace transform. Nieuw Arch. Wisk. (3) **7** (1959), 20–26.

A measurable function $L(x)$, $x>0$, is called a slowly oscillating function (=s.o.f.) if $L(x)>0$ and if, for every $p>0$, $L(px)/L(x)\to 1$ as $x\to\infty$. The author discusses some properties of such functions and proves in particular that to every s.o.f. $L(x)$ there corresponds another such function, $L^*(x)$, so that $L(x)L^*(xL(x))\to 1$ and $L^*(x)L(xL^*(x))\to 1$ as $x\to\infty$. The function L^* is determined uniquely up to asymptotic equivalence, and the relation between L and L^* is symmetric. L and L^* are called a pair of conjugate s.o. functions (c.s.o.f.). The two principal results of the paper are as follows.

Theorem 2: Let A, B, β be real constants, $(1-\beta)A>0$, $\beta(1-\beta)B>0$. Let $P(u)$ be a real function, $\in L(0, R)$ for every $R>0$, and such that $f(s)=s\int_0^\infty P(u)e^{-Asu}du$ converges for every $s>0$. Let L, L^* be c.s.o.f. Then

$$(1) \qquad \log P(u) \sim Bu^\beta[L(u^\beta)]^{(\beta-1)/\beta} \quad \text{as } u^\beta\to\infty$$

implies

$$(2) \quad \log f(s) \sim (1-\beta)B\left(\frac{As}{B\beta}\right)^{\beta/(\beta-1)}[L^*(s^{\beta/(\beta-1)})]^{1/\beta} \quad \text{as } s^{\beta/(\beta-1)}\to\infty;$$

and if $P(u)$ is monotonic, then (2) implies (1).

Theorem 3: Let a, α be positive constants, and let $f(t)$ be a positive measurable function for $0<t\leq a$. Assume that $\inf[f(t): \delta\leq t\leq a]>0$ for each $\delta>0$, and put $F(x)=\int_0^a e^{-xf(t)}dt$ for $x>0$. Let L, L^* be c.s.o.f. Then (3) $f(t)\sim t^\alpha[L(t^{-\alpha})]^{-1}$ as $t\downarrow 0$ implies (4) $F(x)\sim\Gamma(1+\alpha^{-1})[xL^*(x)]^{-1/\alpha}$ as $x\to\infty$; and if $f(t)$ is non-decreasing then (4) implies (3).

These theorems are extensions of results of Kohlbecker [Trans. Amer. Math. Soc. **88** (1958), 346–365; MR **20** #2309] and Békéssy [Magyar Tud. Akad. Mat. Kutató Int. Közl. **2** (1957), 105–120; MR **20** #4134].

A. Erdélyi (Pasadena, Calif.)

5848:

Wyman, Max. The asymptotic behaviour of the Laurent coefficients. Canad. J. Math. **11** (1959), 534–555.

The author has made a deep investigation of the problem stated in the title. We quote the Central Theorem: "A complete asymptotic expansion of the Laurent coefficients of any generating function $G(z)$ that is admissible can be obtained by the formulae established in this section."

The author's term "admissible" covers a wide class of functions, including the Bessel Functions (using his method on this example the author describes as "a little like shooting sparrows with cannons"). For the generating function

$$\exp[\tfrac{1}{2}(1-z^2)]/(1-z)$$

he obtains:

$$a_n \sim \frac{e^{n^{1/2}+1/4}}{(2\pi)^{1/2}n^{1/4}}\left(1+\left(\frac{5}{12}+\frac{(-1)^n}{4}\right)\frac{1}{n^{1/2}}+\cdots\right).$$

For the generating function $\exp(z/(1+z^2))$ he obtains:

$$a_n \sim \frac{2e^{n^{1/2}}\cos[n^{1/2}-\frac{1}{8}(4n-1)\pi]}{(2\pi)^{1/2}(2n)^{3/4}}.$$

S. Chowla (Boulder, Colo.)

FOURIER ANALYSIS

See also 5711a.

5849:

Weiss, Mary; and Zygmund, Antoni. A note on smooth functions. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 52–58.

Les auteurs démontrent les théorèmes suivants. (1) Si $F(x)$ est périodique, et s'il existe un $\beta>\frac{1}{2}$ tel que

$$\Delta^2F(x, h) = F(x+h)+F(x-h)-2F(x) = O\left\{\frac{h}{|\log h|^\beta}\right\},$$

uniformément par rapport à x, F est l'intégrale indéfinie d'une fonction f appartenant à chaque L^p. Ce théorème permet de simplifier un théorème de Salem [mêmes Proc. **57** (1954), 550–555; MR **17**, 845] rendant une de ses hypothèses superflue. (2) Soit F une fonction périodique et continue satisfaisant à la condition $\Delta^2F(x,h)=o\{h/\log h\}$, et soient s_n et σ_n respectivement la somme partielle et la moyenne $(C,1)$ de la série de Fourier de F dérivée terme-à-terme. On a $\lim(s_n-\sigma_n)=0$ uniformément par rapport à x. Les auteurs remarquent que le théorème (1) cesse d'être vrai si $\beta=\frac{1}{2}$. On généralise le théorème (2) dans la métrique L^p.

S. Mandelbrojt (Paris)

5850:

Kumari, Sulaxana. On the Cesàro means of a function. Proc. Math. Phys. Soc. Egypt. no. 22 (1958), 17–25 (1959). (Arabic summary)

Let $f(x)$ be a Lebesgue integrable function. Let

$$f(x)\sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{A_0}{2}+\sum_{n=1}^{\infty}A_n(x).$$

Write

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)-2s\},$$

$$\varphi_\alpha(t)=\frac{\alpha}{t^\alpha}\int_0^t(t-u)^{\alpha-1}\varphi(u)du\qquad(\alpha>0).$$

Let a_n, b_n and $C_\alpha(\omega)$ be respectively the Fourier coefficients and the Riesz means of order α of $f(x)$, where

$$C_\alpha(\omega)=\omega^{-\alpha}S_\alpha(\omega),$$

$$S_\alpha(\omega)=\sum_{n<\omega}(\omega-n)^\alpha A_n(x).$$

The author generalizes the results of Loo [Trans. Amer. Math. Soc. **56** (1944), 508–518; MR **6**, 126] and Hyslop [J. London Math. Soc. **24** (1949), 91–100; MR **11**, 100] as follows.

(A) If, for $\alpha>-1$, $1>\rho>-1$, $\alpha>\rho$, $-1\leq p<\infty$,

(a) $$C_\alpha(\omega)-s=o(\omega^{-\rho}(\log\omega)^p)\quad(\omega\to\infty),$$

then

(b) $$\varphi_{\alpha+1-\rho}(t)=o(t^\rho(\log t^{-1})^{p+1})$$

as $t\to 0$.

(B) If $\alpha>-1$, $1>\rho>-1$, $0\geq\alpha-\rho>-1$, $-1\leq p<\infty$ and $a_n,b_n=O(n^{\alpha-\rho-\delta})$ $(\delta>0)$, then (a) implies (b).

Fu Cheng Hsiang (Taipeh)

5851:

★Bochner, Salomon. Lectures on Fourier integrals. With an author's supplement on monotonic functions, Stieltjes integrals, and harmonic analysis. Translated by Morris Tenenbaum and Harry Pollard. Annals of Mathematics Studies, No. 42. Princeton University Press, Princeton, N.J., 1959. viii+333 pp. $5.00.

From the translators' preface: "In undertaking this translation of Bochner's classical book [*Vorlesungen über Fouriersche Integrale*, Akad. Verlag., Leipzig, 1932] and its supplement [Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse, Math. Ann. **108** (1933), 378–410], our main purpose was to make generally available to the present generation of group-theorists and practitioners in distributions the historical and concrete problems which gave rise to these disciplines. Here can be found the theory of positive definite functions, of the generalized Fourier integral, and even forms of the important theorems concerning the reciprocal of Fourier transforms."

5852:

Tsuji, Kazô. ω-almost periodic functions on arbitrary groups. Bull. Kyushu Inst. Tech. Math. Nat. Sci. **4** (1958), 7–14.

Maak has established [J. Reine Angew. Math. **190** (1952), 34–48; MR **13**, 910] the existence of a mean value for ω-almost periodic functions on groups. The author extends this result to functions taking their values in a complete locally convex topological linear space.

K. deLeeuw (Stanford, Calif.)

5853:

★Malliavin, Paul. Impossibilité de la synthèse spectrale sur les groupes abéliens non compacts. Séminaire P. Lelong, 1958/59, exp. 17, 8 pp. Faculté des Sciences de Paris, 1959.

A slightly expanded version is reviewed below [#5854c].

5854a:

Malliavin, Paul. Sur l'impossibilité de la synthèse spectrale dans une algèbre de fonctions presque périodiques. C. R. Acad. Sci. Paris **248** (1959), 1756–1759.

5854b:

Malliavin, Paul. Sur l'impossibilité de la synthèse spectrale sur la droite. C. R. Acad. Sci. Paris **248** (1959), 2155–2157.

5854c:

Malliavin, Paul. Impossibilité de la synthèse spectrale sur les groupes abéliens non compacts. Inst. Hautes Études Sci. Publ. Math. **1959**, 85–92.

The author shows how to construct a counter-example to spectral synthesis in each non-compact, locally compact, abelian group. The result had been ardently sought for several years for the additive group of real numbers; no progress at all had been made until this strikingly original work appeared.

Let Γ be a locally compact abelian group and $L^1(\Gamma)$ its group algebra. We denote by G the character group of Γ. It is well known that if Γ is compact then every closed ideal I in $L^1(\Gamma)$ has the form $I=I(S)$, where for a closed subset S of G, $I(S)$ is the ideal of all functions in $L^1(\Gamma)$ whose Fourier transforms vanish on S. The author shows that whenever Γ is non-compact there exists a closed ideal I which is not of this form. In fact, as Walter Rudin has remarked, the proof shows that for each natural number n there is an ideal I such that $I^{n+1}\neq I^n$, where I^k denotes the smallest closed ideal containing k-fold convolutions of functions in I.

The question is easily reduced to the case when Γ is discrete, which is technically easier to describe. One may

identify $L^1(\Gamma)$ with $A(G)$, the algebra of continuous functions on G with absolutely convergent Fourier series. Let $A'(G)$ be the dual Banach space to $A(G)$, so that if $f \in A'(G)$ we have a linear functional $\langle \varphi, f \rangle$ defined for $\varphi \in A(G)$. The norm of f is easily seen to be $N(f) = \sup|\langle \chi, f \rangle|$, where χ ranges over all the continuous characters of G. $A'(G)$ is obviously an $A(G)$-module. The spectrum (support) of an element $f \in A'(G)$ is a closed subset $S(f)$ of G defined by saying that $x \notin S(f)$ if there exists a $\psi \in A(G)$ with $\psi(x) \neq 0$ such that $\psi f = 0$. The problem at hand is to decide whether for $\varphi \in A(G)$ the vanishing of φ on $S(f)$ implies that $\varphi f = 0$. The author's procedure is to give an explicit construction of a φ and an f with $\varphi = 0$ on $S(f)$ such that $\langle \varphi, f \rangle \neq 0$.

When the dimension of G is at least 3, Laurent Schwartz [C. R. Acad. Sci. Paris **227** (1948), 424–426; MR **10**, 249] had earlier obtained a counter-example. His procedure will be outlined briefly. Let S be a suitable hypersurface in G, say a sphere, μ an appropriate non-trivial Radon measure concentrated on S, and D the operation of differentiation with respect to some coordinate. We define f by $\langle \varphi, f \rangle = \int_S D\varphi(x)\mu(dx)$. It is easy to see that $S(f) = S$, but one can have $\varphi \equiv 0$ on S while $\int_S D\varphi(x)\mu(dx) \neq 0$. The essential matter is to arrange things so that $f \in A'(G)$. This cannot be done if $\dim G < 3$; however, the procedure warrants closer inspection in a slightly different form. Any continuous function g on G defines an element in $A'(G)$ by $\langle \varphi, g \rangle = \int_G \varphi(x)g(x)dx$, where the integration is with respect to the Haar measure. Let $\{g_n\}$ be a sequence of functions such that g_n vanishes outside the open set S_n, where $\bigcap S_n = S$, and such that for each $\varphi \in A(G)$, $\lim_{n\to\infty} \int \varphi(x)g_n(x)dx = \int \varphi(x)\mu(dx)$. For a suitable choice of the sequence $\{g_n\}$, $f = -\lim_{n\to\infty} Dg_n$, where the limit is taken in the w^*-topology of $A'(G)$. The crucial matter, of course, is to make sure that $N(Dg_n)$ is uniformly bounded.

The present author looks at Schwartz's idea in the following way. Suppose S is the set of zeros of a real-valued function $\xi \in A(G)$. Take $\{G_n\}$, a sequence of smooth functions of a real variable such that G_n vanishes outside $(-n^{-1}, n^{-1})$, say, and $\int_{-\infty}^{\infty} G_n(\xi)d\xi = 1$. Take $g_n(x) = G_n(\xi(x))$; then, at least formally, $\{g_n\}$ converges weakly in $A'(G)$ to a measure on S which ought to be written $\delta(\xi)$. The novelty is the modification of the operator D. Differentiation with respect to a coordinate is useless if $\dim G < 3$ and does not even make sense if $\dim G = 0$. Instead, the author takes $D = \partial/\partial\xi$. The resulting $f = -\lim_{n\to\infty} Dg_n$ could be written as $f = -\delta'(\xi)$. What has to be done is to choose ξ so that everything makes sense; the heuristic reasoning discussed above is decidedly only half the genuine content of the present work.

One can represent δ and δ' by means of the Fourier transforms

$$\delta(\xi) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-iu\xi)du,$$

$$\delta'(\xi) = (2\pi i)^{-1}\int_{-\infty}^{\infty} \exp(-iu\xi)udu.$$

In order that $\delta'(\xi)$ belong to $A'(G)$ we must have

$$\langle \chi, \delta'(\xi) \rangle = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\{\int_G \chi(x)\exp\{-iu\xi(x)\}dx\right\}udu$$

uniformly bounded over $\chi \in \Gamma$. Suppose $\xi(x) = a + \sum a_\chi \bar{\chi}(x)$, where $\sum |a_\chi| < \infty$, the sum extending over the non-trivial characters $\chi \in \Gamma$. It suffices to prove that $\int_0^\infty |F(\chi, u)|udu$ is uniformly bounded, where

$$F(\chi, u) = \int_G \chi(x)\exp\{-iu\sum a_\varphi \bar{\varphi}(x)\}dx.$$

The counter-example is obtained by noting that $\xi = 0$ on $S(\delta'(\xi))$, but

$$-\langle \xi, \delta'(\xi) \rangle = \langle 1, \delta(\xi) \rangle$$
$$= (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(-iau)F(\chi_0, u)du,$$

χ_0 being the trivial character. $F(\chi_0, u)$ is a continuous function of u which has the value 1 when $u = 0$. Hence its Fourier transform cannot vanish identically; i.e., for some choice of a, $\langle \xi, \delta'(\xi) \rangle \neq 0$.

The construction of ξ in the general case is too complicated to discuss here. Most of the main ideas can be seen in the situation where G is the countable direct sum of groups of order 2. One represents $x \in G$ as a sequence $x = (x_1, x_2, x_3, \cdots)$ where $x_n = 0$ or 1. Define $\chi_n(x) = (-1)^{x_n}$. The most general character is of the form χ_N, where N is a finite set of natural numbers and $\chi_N = \Pi_{n\in N}\chi_n$. One considers ξ's of the form $\xi = a + \sum a_n\chi_n$; in the general case one has to do with lacunary series. For this choice of ξ the integral defining $F(\chi, u)$ decomposes into a product and we get

$$F(\chi_N, u) = \Pi_{n\in N}(-i\sin a_n u)\Pi_{m\notin N}(\cos a_m u).$$

Such products may be estimated in the following manner {the author's estimates are clumsy; elementary techniques yield far more powerful results than the facts about Bessel functions which he invokes}: for $0 \leqq v < 1/2$ one has $|\sin v|, |\cos v| \leqq \exp(-v^2/3)$. Thus

$$|F(\chi_N, u)| \leq \exp\{-(u^2/3)\textstyle\sum^* a_n^2\},$$

where $\sum^*$ denotes the sum over those n for which $|a_n u| < \frac{1}{2}$. For example, if $a_n = n^{-5/4}$, $\sum^* a_n^2 = O(u^{-3/2})$. Thus $|F(\chi_N, u)| = O(\exp(-u^{1/2}))$. In the general case one can always obtain uniform estimates of the form

$$|F(\chi, u)| = O(\exp(-\omega(u))),$$

where ω is concave increasing with $\int_1^\infty u^{-2}\omega(u)du < \infty$.

The last remark of the above paragraph shows that for suitable ξ, $\delta^{(n)}(\xi) \in A'(G)$ for all n, whence Rudin's remark. However, something else is available. Suppose F is a function of a real variable such that $F(\varphi) \in A(G)$ for all real functions $\varphi \in A(G)$. We may then note that $\langle F(\xi), \delta^{(n)}(\xi) \rangle$ exists. This expression is $\pm \langle d^nF/d\xi^n, \delta(\xi) \rangle$. One has sufficient freedom in the choice of ξ to conclude that $d^nF/d\xi^n$ must exist at $\xi = 0$. Indeed, one can obtain bounds of quasi-analytic type for the derivatives of F at the origin. This gives something quite close to the results of Helson, Kahane, Katznelson, and Rudin [Acta Math. **102** (1959), 135–157]. It was just such considerations, on the part of Kahane and Katznelson, which put Malliavin on the track to his result. *C. S. Herz* (Ithaca, N.Y.)

5855:

Kahane, Jean-Pierre. Sur un théorème de Paul Malliavin. C. R. Acad. Sci. Paris **248** (1959), 2943–2944.

Let the notation be as in the review above. In the case Γ is the additive group of integers a result of Beurling and

Pollard, stated for the real line by Pollard [Duke Math. J. **20** (1953), 499–512; MR **15**, 215], asserts that if $\varphi \in A(G)$, $f \in A'(G)$, φ satisfies a Lipschitz condition of order 1/2 in a neighborhood of $S(f)$ and $\varphi=0$ on $S(f)$, then $\langle\varphi, f\rangle=0$. This theorem admits certain generalizations which lead one to suspect that the order 1/2 is best possible. In the construction of Malliavin the function ξ never satisfies a Lipschitz condition of positive order. In the present paper Kahane takes $\xi=a+\sum a_\chi\chi$ where the a_χ's are random coefficients. With the proper choice of distributions one finds that ξ satisfies the condition of Malliavin and a Lipschitz condition of order $\alpha>1/5$ with probability 1. Then for suitable constant terms one has $\langle\xi, \delta'(\xi)\rangle\neq 0$. It does not seem likely that the method of the paper can be refined to give an $\alpha>1/4$.

C. S. Herz (Ithaca, N.Y.)

INTEGRAL TRANSFORMS AND OPERATIONAL CALCULUS

See also 5719, 5847, 5888.

5856:

Vasilach, Serge. Sur quelques extensions de la théorie des moments bornés. C. R. Acad. Sci. Paris **249** (1959), 610–612.

In an earlier paper [same C. R. **246** (1958), 676–678; MR **20** #2581] the author gave an extension of Phragmén's theorem. Making use of that extension he derives two theorems dealing with an extension of the theory of bounded moments for Lebesgue integrable functions whose values lie in a Banach space.

H. P. Thielman (Ames, Iowa)

5857:

Vasilach, Serge. Sur quelques extensions de la théorie de moments bornés. C. R. Acad. Sci. Paris **249** (1959), 813–815.

This is an extension of an earlier result by the author. (See preceding review.) *H. P. Thielman* (Ames, Iowa)

5858:

Fox, Charles. An application of fractional integration to chain transform theory. Proc. Amer. Math. Soc. **9** (1958), 968–973.

A chain transform of order n, as previously defined by the author, is in substance a set of n transforms $T_1, T_2, \cdots, T_n$ such that $T_n=T_{n-1}T_{n-2}\cdots T_1$; the T's are of the form

$$Tg = G(x) = \int_0^\infty k(u, x)g(u)du$$

or of the integrated form; the kernel $k(u, x)$ is of the type (i) $l(ux)$ or (ii) $u^{-1}r(x/u)$. Here only the case is considered when all the T's are of the type (ii), e.g., $r_1(v)=v^{-1}e^{-1/v}$, $r_2(v)=e^{-v}$, $r_3(v)=(1+v)^{-1}$; $n=3$. Let $I_{\eta,\alpha}f$ be the fractional integral

$$\{x^{\eta+\alpha}\Gamma(\alpha)\}^{-1}\int_0^x (x-t)^{\alpha-1}t^\eta f(t)dt \quad (\Re(\alpha) > 0, \Re(\eta) > 0).$$

It is shown that, if $r_j^*(v)=I_{\alpha_j,\eta_j}r_j$, then the r_j^* ($j=1, 2, \cdots, n$) are the kernels of another chain transform, provided that the Mellin transforms of the I's satisfy a certain condition. {Reviewer's remark: This condition, however, can be considerably simplified; when $n=3$, for instance, the I's must be of the form $I_{\eta,\alpha}, I_{\eta+\alpha,\beta}$ (or $I_{\eta+\alpha,\beta}, I_{\eta,\alpha}$) and $I_{\eta,\alpha+\beta}$, respectively.}

H. Kober (Birmingham)

5859:

Tanno, Yûkichi. On the convolution transform. Kōdai Math. Sem. Rep. **11** (1959), 40–50.

Let

$$F(s) = \prod_{k=1}^\infty [1-s^2/a_k^2][1-s^2/c_k^2]^{-1},$$

where $0<a_1\leq a_2\leq\cdots$, $0<c_1\leq c_2\leq\cdots$, $a_k<c_k$, and $\lim_{n\to\infty}n/a_n=\Omega>\Omega'=\lim_{n\to\infty}n/c_n$, and let

$$G(t) = (2\pi i)^{-1}\int_{-i\infty}^{i\infty} [F(s)]^{-1}e^{st}ds.$$

The author studies the convolution transform $f(x)=\int_{-\infty}^\infty G(x-t)\phi(t)dt$ and its conversion by integro-differential operators—thus generalizing some of the results in *The convolution transform* by D. V. Widder and the reviewer [Princeton Univ. Press, 1955; MR **17**, 479]. The demonstrations given for some of the principal results of the present paper are inadequate.

I. I. Hirschman, Jr. (St. Louis, Mo.)

5860:

Blackman, Jerome; and Pollard, Harry. The finite convolution transform. Trans. Amer. Math. Soc. **91** (1959), 399–409.

Let $f(x)=\int_0^x \phi(x-t)dk(t)$, where f and k are given and ϕ is to be determined. The case $0\leq x<\infty$ is classical; in this paper the authors deal with the case $0\leq x<a$.

Let z_i $(=x_i+iy_i)$ be a zero of the entire function $\hat{k}(s)$ given by $\hat{k}(s)=\int_0^{2b} e^{-su}dk(u)$; $0\leq x<b<a/2$. Then the authors decompose these zeros into classes A_n, $n=0, 1, 2, \cdots$, depending upon the value of $(y_i^2-x_i^2)^{1/2}$ and also associate with each zero z a function $h_z(x)$, $0\leq x<\infty$, defined as follows:

$$h_z(x) = |z|^2\int_0^x e^{z(x-t)}e^{\bar{z}t}dt, \ z \text{ imaginary},$$

$$h_z(x) = -ze^{zx}, \ z \text{ real},$$

where $\bar{z}$ is the complex conjugate of z. Let the convolution of $l(x)$ and $m(x)$ be denoted by l^*m, where

$$l^*m = \int_0^x l(t)m(x-t)dt,$$

and let Π^* denote the convolution of several such functions. Then the authors define the functions $H_k(x)$, $k=0, 1, 2, \cdots$, as follows:

$$H_0(x) = \lim_{m\to\infty} f(x)^* \prod_{i=1;z_i\in A_0}^{m}{}^* h_{z_i}(x),$$

$$H_k(x) = \lim_{m\to\infty} H_{k-1}(x)^* \prod_{i=1;z_i\in A_k}^{m}{}^* h_{z_i}(x).$$

The authors then prove that $\lim_{k\to\infty}H_k(x+c)=B\phi(x)$, $0\leq x\leq b$, where c and B are known constants. This gives the solution for $\phi(x)$ when $0\leq x\leq b$.

To complete the solution the authors then show that the

solution for the case $0 \leq x \leq (n+1)b$ can be derived from that of $0 \leq x \leq nb$ by means of some simple changes of variables. *C. Fox* (Montreal, P.Q.)

5861:

★Doetsch, Gustav. Einführung in Theorie und Anwendung der Laplace-Transformation. Ein Lehrbuch für Studierende der Mathematik, Physik und Ingenieurwissenschaft. Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften. Mathematische Reihe Bd. 24. Birkhäuser Verlag, Basel-Stuttgart, 1958. 301 pp. DM 39.40.

This book represents, in the opinion of the reviewer, the best treatment of the one-dimensional Laplace transformation in the literature. Its modest purpose is to bridge the gap between purely formal treatments of the subject and complete mathematical discussions found in monographs and papers. Because of his long and intimate connection with the Laplace transformation, the author has written much more than another text on the subject. Each topic is covered with a completeness and sharpness of focus that is unusual. There is a wealth of illustrative detail, but there is no sacrifice of mathematical precision and all proofs are clearly and forcefully given. The list of topics is not unusual; it includes the basic properties of the transformation, its application to ordinary differential equations, its application to difference and integral equations, various forms of the inversion integral, the asymptotic behavior of the transform and of the original function. But the fact that all this, together with the necessary material from complex function theory, theory of ordinary differential equations, and Fourier transforms, is given in such a small book, emphasizes the excellence of the exposition. Every sentence is significant and illuminating, and there are no unnecessary or irrelevant remarks.

D. L. Bernstein (Baltimore, Md.)

5862:

★Doetsch, Gustav. Introduction à l'utilisation pratique de la transformation de Laplace. Traduit de l'allemand par M. Parodi. Avec un appendice: Tables de correspondances, par R. Herschel. Gauthier-Villars, Paris, 1959. viii + 198 pp. Paperbound: 3500 francs; \$8.55.

The original [*Anleitung zum praktischen Gebrauch der Laplace-Transformation*, R. Oldenbourg, München, 1956] is reviewed in MR **19**, 139.

5863:

★Holl, Dio L.; Maple, Clair G.; and Vinograde, Bernard. Introduction to the Laplace transform. The Appleton-Century Mathematics Series. Appleton-Century-Crofts, Inc., New York, 1959. viii + 174 pp. \$4.25.

Textbook for undergraduates in engineering, with illustrative applications and brief tables.

5864:

Delavault, Huguette. Détermination d'une fonction $F(t)$ dont on connaît la transformée de Laplace en une infinité de points. Application. C. R. Acad. Sci. Paris **247** (1958), 1284–1287.

The author shows that if $f(\alpha_p) = f_p$ with $\operatorname{Re} \alpha_p > 0$, $\alpha_p \to \infty$ as $p \to \infty$, $\alpha_p^\gamma f_p$ bounded, then the necessary and sufficient condition that $f(s)/s$ (or $f(s)$ if $\gamma > 1$) be uniquely determined and be a Laplace transform of some function $F(t)$ is that

$$\sum [1 - (|\alpha_p - \tfrac{1}{2}| + |\alpha_p + \tfrac{1}{2}|)]/(1 - r_p)$$

diverges, where r_p are successively determined by $\alpha_1, \cdots, \alpha_p, f_1, \cdots, f_p$. It is shown that $F(t)$ is the derivative of the limit of an absolute value of certain line integral. These results are applied to a function of two variables defined by a differential equation of elliptic type in the half-infinite strip bounded by rays R_1, R_2, and a segment S. Assuming that $F(x, 0)$, $F(0, t)$ (on R_1 and S) as well as $\partial F(x, 0)/\partial t$ are given, the values of F on R_2 are determined.

Č. Masaitis (Havre de Grace, Md.)

5865:

Fenyő, István. Über die Verallgemeinerung der Operatorenrechnung. Publ. Math. Debrecen **6** (1959), 48–59.

L'A. considère l'anneau des fonctions continues sur un rectangle et nulle en dehors, avec le produit de convolution $U_* V = \int_x^y U(x, t) V(t, y) dt$. Il démontre que les éléments permutables avec un élément fixe U, tel que $U(x, x) \neq 0$ et ayant les dérivées continues $\partial U/\partial x$, $\partial U/\partial y$, $\partial^2 U/\partial x \partial y$, forment un sous-anneau sans diviseurs de zéro, isomorphe à l'anneau considéré par Mikusiński, qui correspond à la fonction unité. Par extension algébrique, ou par le procédé de Korevaar, on obtient alors des corps isomorphes à celui de Mikusiński.

G. Marinescu (Bucharest)

5866:

Jain, Mahendra Kumar. On certain symbolic operators. Gaṇita **9** (1958), 77–82.

Burchnall [Quart. J. Math. **3** (1932), 213–223] derived a number of formulae involving the operator $\delta = x(d/dx)$. In particular he showed that the symbolic form of the Laplace transform $F(x) = \int_0^\infty e^{-xt} f(t) dt$ is

$$F(x) = x^{-1} \Gamma(1-\delta) f(1/x).$$

The author uses Burchnall's methods to show that the Whittaker transform can be expressed in terms of three operations involving Laplace transforms.

J. L. Griffith (Kensington)

5867:

Koizumi, Sumiyuki. On the singular integrals. IV. Proc. Japan Acad. **34** (1958), 653–656.

The author continues his work, done on Hilbert's operator

$$\tilde{g}(x) = \lim_{\eta \to 0} \int_{|x-t| > \eta > 0} g(t)(x-t)^{-1} dt = \mathfrak{H} g$$

in part III [same Proc. **34** (1958), 594–598; MR **21** #2876]. Dealing with the case $g(t) \in L_\alpha^p$, defined by

$$\int_{-\infty}^{\infty} \frac{|g(t)|^p}{1+|t|^\alpha} dt < \infty \quad (p \geq 1,\ 0 \leq \alpha < 1),$$

he proves (theorem 5), if $g(x) \in L_\alpha^p$ and $p > 1$, or if both $g(x)$ and $\tilde{g}(x) \in L_\alpha^1$, then $\mathfrak{H}\tilde{g} = -g$. He uses a complex variable method, extending the Hille-Tamarkin results [Fund. Math. **25** (1935), 329–352] on the class $\mathfrak{H}^p$ to the class $\mathfrak{H}_\alpha^p$ which is defined as follows: $f(z)$ $(z = x+iy)$ is analytic for $y > 0$ and $\int_{-\infty}^{\infty} |f(x+iy)|^p (1+|x|^\alpha)^{-1} dx$

$<$ const for $0<y<\infty$, so that $\mathfrak{H}_0{}^p=\mathfrak{H}^p$. The representation of a function $f(z)\in\mathfrak{H}_\sigma{}^p$ $(p\geqq 1)$ both by its Cauchy and its Poisson integral is investigated, and the Blaschke product associated with $f(z)$ is deduced also.

It is, however, known that, even for $\alpha=p=1$, i.e., when $f(z)(z+i)^{-1}\in\mathfrak{H}^1$, $f(z)$ is representable by its Cauchy integral; and that, if both $g(x)$ and $\tilde{g}(x)$ belong to $L_1{}^1$, $\mathfrak{H}\tilde{g}=-g$: which incidentally implies theorems 5 and 6 of the reviewer, Amer. J. Math. **68** (1946), 398–416 [MR **8**, 152], see Lemmas 9 and 11. *H. Kober* (Birmingham)

5868:

Koizumi, Sumiyuki. On the singular integrals. V. Proc. Japan Acad. **35** (1959), 1–6.

A generalised Hilbert transform of order r is introduced,

$$f_r(x)=\frac{(x+i)^r}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t+i)^r}\frac{dt}{x-t}=\mathfrak{H}_r f \qquad (r=0,1,2,\cdots),$$

so that $f_0(x)=\mathfrak{H}f$, $f_r(x)(x+i)^{-r}=\mathfrak{H}[f(t)(t+i)^{-r}]$. Its integrability is discussed, and from previous results a considerable number of formulae are obtained, for instance

$$(2.4)\quad \int_{-\infty}^{\infty}\frac{|f_r(x)|^p}{1+|x|^{rp}}\frac{dx}{1+|x|^\alpha}\leqq f_{p,\alpha}\int_{-\infty}^{\infty}\frac{|f(x)|^p}{1+|x|^{rp}}\frac{dx}{1+|x|^\alpha} \qquad (p>1,\ 0\leqq\alpha<1).$$

Then discrete sequences $(\cdots x_{-1}, x_0, x_1, \cdots)$ are dealt with in a similar way, by the transformation

$$(3.2)\quad \tilde{x}_n{}^{(r)}=(n+i)^r\sum_{\substack{m=-\infty\\ m\neq n}}^{\infty}\frac{x_m}{(m+i)^r}\frac{1}{n-m} \qquad (n=\cdots,-1,0,1,\cdots),$$

and finally the reciprocal formula and analytic functions in the upper half plane [cf. part IV above] are discussed in connection with the above generalisation.

H. Kober (Birmingham)

5869:

Koppelman, W.; and Pincus, J. D. Spectral representations for finite Hilbert transformations. Math. Z. **71** (1959), 399–407.

Let a be a sub-interval of $[-\infty,\infty]$. If $x\in L^2(a)$, then $T_a x$ denotes the function g defined by the relation

$$\pi i\cdot g(\theta)=\int_a x(\mu)\cdot(\mu-\theta)^{-1}d\mu$$

(Cauchy principal value for any θ in a). If $a=[-\infty,\infty]$, then T_a is the usual Hilbert transformation; this article concerns the case $a\neq[-\infty,\infty]$. Let $|a|$ denote the length of a. In case $|a|=\infty$, it is shown that T_a is isometrically equivalent to an operator T_b, where $|b|\neq\infty$. A suitable isometry between $L^2(b)$ and $L^2[-1,1]$ causes that T_b is isometrically equivalent to the unit-multiplicator $\{f\to I^1\cdot f\}$ (where I^1 is the unit-function defined by $I^1(\theta)=\theta$ for $|\theta|\leqq 1$). Thus, the spectral representation for T_a can be explicitely obtained, and the spectrum of T_a is the interval $[-1,1]$. The authors also determine a family of eigenfunctions for T_a whose corresponding eigenvalues lie in the interior of $[-1,1]$. In case $|a|\neq\infty$, a suitably defined differential operator D_a is found to coincide with $f(T_a)$, where f is the function defined by $2\pi\cdot f(\theta)=|a|\log[(1-\theta)\cdot(1+\theta)^{-1}]$ for all θ in $[-1,1]$.

G. L. Krabbe (Lafayette, Ind.)

5870:

Rajagopal, A. K. A general transform—Gauss transform. Proc. Indian Acad. Sci. Sect. A **50** (1959), 143–148.

Let $D=d/dx$; $A(x)=ax^2+bx+c$;

$$\log\omega(x)=-\int^x(pt+q)/A(t)dt,$$

where a, b, c, p and q are constants independent of n. The author has previously proved that

$$p_n(x)=D^n[\{A(x)\}^n\cdot\omega(x)]/K_n\omega(x),$$

where K_n is a constant, are orthogonal polynomials for some interval (α,β). The author defines the operators T_1 and T_2 as follows:

$$f_1(n)=T_1f(x)=\int_\alpha^\beta\omega(x)f(x)p_n(x)dx,$$

$$f_2(n)=T_2f(x)=\int_\alpha^\beta f(x)p_n(x)dx.$$

He then proves that if $\mathrm{lt.}_{x\to\alpha,\beta}A(x)\omega(x)f(x)=0$ and $\mathrm{lt.}_{x\to\alpha,\beta}A(x)\omega(x)f'(x)=0$ then

$$T_1[D\{A(x)\omega(x)f'(x)\}/\omega(x)]=-n[p-(n+1)a]f_1(n),$$

and that if $\mathrm{lt.}_{x\to\alpha,\beta}A(x)f(x)=0$ and $\mathrm{lt.}_{x\to\alpha,\beta}A(x)f'(x)=0$ then

$$T_2[D\{A(x)f'(x)+(px+q)f(x)\}]=(px+q)f(x)p_n(x)|_\alpha{}^\beta-n[p-(n+1)a]f_2(n).$$

The author also obtains a formula for the inverse operator $T_1{}^{-1}$. *C. Fox* (Montreal, P.Q.)

5871:

Rao, V. V. L. N. On self reciprocal functions involving hypergeometric functions. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 176–183.

The author obtains some examples, involving hypergeometric functions, of functions self reciprocal with respect to Hankel transformations of various orders.

P. G. Rooney (Toronto, Ont.)

5872:

Rao, V. V. L. N. A few self-reciprocal functions. Proc. Indian Acad. Sci. Sect. A **50** (1959), 206–212.

The author finds some functions which are their own Hankel transforms. *P. G. Rooney* (Toronto, Ont.)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

5873:

Hvedelidze, B. V. Linear discontinuous boundary problems in the theory of functions, singular integral equations and some of their applications. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **23** (1956), 3–158. (Russian)

Let Γ denote a set of one or more suitably smooth curves in the complex plane, and S the integral operator given by

$$S\phi = (\pi i)^{-1}\int_\Gamma \phi(t)(t-t_0)^{-1}dt,$$

mapping one complex-valued function into another. The three chapters of this book deal with (i) properties of S and of integral equations involving it, (ii) applications to the "Riemann-Privalov" boundary problem $\Phi^+(t) = a(t)\Phi^-(t)+b(t)$, where $a(t)$, $b(t)$ are given on Γ, and $\Phi^+(t)$, $\Phi^-(t)$ are the boundary values, on the two sides of Γ, of analytic functions to be found, (iii) applications to other boundary problems, previously studied by Hilbert, Riemann and Vekua, and Poincaré.

The already well-developed theory of these matters is expounded in the books of N. I. Mushelišvili [*Nekotorye osnovnye zadači matematičeskoĭ teorii uprugosti*, Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1933, 1935, 1949, 1954; *Singulyarnye integral'nye uravneniya*, OGIZ, Moscow-Leningrad, 1946; translated as *Some basic problems of the mathematical theory of elasticity* and *Singular integral equations*, Noordhoff, Groningen, 1953; MR **11**, 626; **16**, 1067; **8**, 586; **15**, 370, 434]. However, in these books and in much of the literature, Hölder or similar conditions are imposed on the values on Γ of the functions occurring. The keynote of the present book is the use of Lebesgue classes, permitting the author to give the theory a more complete form.

Basic in Chapter I is the class $L_p(\Gamma;\rho)$ of functions $f(t)$ measurable on Γ, and such that $\rho(t)|f(t)|^p$ is summable on Γ; here $\rho(t)$ is non-negative on Γ, and subject to $\int_\Gamma \rho^{1-q}(t)ds<\infty$, where $p^{-1}+q^{-1}=1$ and ds is the element of arc. There is a detailed study of the boundedness and the inversion of S in the space $L_p(\Gamma;\rho)$, mainly in the case

$$\rho(t) = \prod_1^{m_1}|t-c_k|^{\alpha_k(p-1)}\prod_{m_1+1}^{m}|t-c_k|^{-\alpha_k},$$

where $0\leq\alpha_k<1$, and the c_k are points of Γ, often end-points of constituent arcs. The author is thus enabled to discuss the solution in $L_p(\Gamma;\rho)$ of the Noether equation

$$a(t)\phi(t)+b(t)S\phi(t)+V\phi(t) = f(t),$$

where V is completely continuous, utilising results on linear equations in Banach spaces. It is assumed that $a(t)$, $b(t)$ are continuous, this being strengthened to a Dini condition if Γ contains open arcs; the vector-matrix analogue is also considered. The last two sections of Chapter I deal with the representation of analytic functions in the form

$$(2\pi i)^{-1}\int_\Gamma \phi(t)(t-z)^{-1}dt+P(z),$$

where $\phi(t)$ belongs to some Lebesgue class on Γ, and $P(z)$ is an entire function; the two essentially distinct cases are those of a "Cauchy integral", when Γ consists of closed curves only, and of an "integral of Cauchy type", when Γ may include open curves. It should be stressed that this summary can give little idea of the wealth of theorems in the individual sections of this chapter.

As explained in Mushelišvili's *Singular integral equations*, cited above, the Riemann-Privalov problem is soluble by quadratures in the scalar case, assuming that $a(t)$ and $b(t)$ satisfy Hölder conditions, except possibly for a finite number of discontinuities. The present author's treatment involves three extensions, firstly that the Hölder conditions on $a(t)$, $b(t)$ are weakened to Dini conditions [see however the review below], secondly that $\Phi^+(t)$, $\Phi^-(t)$ need not be continuous on the boundary Γ, but must however be representable by Cauchy integrals, or integrals of Cauchy type, with integrands satisfying a Lebesgue condition, and thirdly that the boundary relation is to hold almost everywhere, instead of everywhere. The case in which $b(t)$ belongs to a Lebesgue class is also considered. The case when $a(t)$ is a square matrix, and Φ^+, Φ^-, b are vectors, is not generally soluble by quadratures completely, since difficulties arise in the solution of the homogeneous equation ($b(t)\equiv 0$). The author relies on results of N. P. Vekua [*Sistemy singulyarnyh integral'nyh uravnenii i nekotorye graničnye zadači*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950; MR **13**, 247] on the latter point, and so his restrictions on $a(t)$, $b(t)$ are more severe than in the scalar case. The chapter closes with applications of the theory of this boundary problem to the solution of the Noether integral equation.

In the last chapter similar extensions are made in the theory of other known boundary problems. There is first the "Riemann-Hilbert" problem of finding $\Phi(z)$, analytic in one of the two regions into which Γ divides the plane, whose real and imaginary parts u, v satisfy $au-bv=c$ on Γ. In addition to the usual condition $a^2(s)-b^2(s)\neq 0$ on Γ, $a(s)$ and $b(s)$ must satisfy a Dini condition on Γ, while $c(s)\in L_p(\Gamma;\rho)$, and $\Phi(z)$ is to be representable by a Cauchy integral. Next considered is a much more general class of boundary problems, involving $\Phi(z)$ and its derivatives, studied by N. P. Vekua, I. N. Vekua [see, e.g., same Trudy **11** (1942), 109–139; MR **6**, 123] and the author. Finally, applications are given to the boundary problem $u_{xx}+u_{yy}+X(x,y)u_x+Y(x,y)u_y+Z(x,y)u=0$ subject to $A(t)u_x+B(t)u_y+C(t)u=F(t)$ for points t on Γ.

Commendable features are the extensive bibliography and the detailed historical reviews of individual problems.

F. V. Atkinson (Canberra)

5874:

Hvedelidze, B. V. A remark on my work "Linear discontinuous boundary problems in the theory of functions, singular integral equations and some of their applications." Soobšč. Akad. Nauk Gruzin. SSR **21** (1958), 129–130. (Russian)

The treatment of the Riemann-Privalov boundary problem in the cited work [see preceding review] involved that $a(t)$ satisfy a Dini condition, restricting the modulus of continuity. This is now relaxed to the simple requirement of continuity. This extension apparently applies only to the scalar case. *F. V. Atkinson* (Canberra)

5875:

Przeworska-Rolewicz, D. Sur l'application de la méthode des approximations successives à une équation intégrale à forte singularité. Ann. Polon. Math. **6** (1959), 161–170.

A real function $G(t,\tau,x,y)$ of the real variables x and y and complex variables t, τ belongs to class H_μ, on a curve L if, for $t,\tau\in L$, $|x|\leq R$, $|y|\leq R$, G can be written as

$$G(t,\tau,x_1,y_1)-G(t,\tau,x_2,y_2) = G_1(t,\tau,x_1,x_2,x_3,x_4)G_2(x_1-x_2)+ G_3(t,\tau,x_1,x_2,x_3,x_4)G_4(x_3-x_4),$$

where $G_2(0)=G_3(0)=0$, G_2 and G_4 satisfy Lipschitz conditions, and G_1 and G_3 satisfy

$$|G_i(t, \tau, x_1, x_2, x_3, x_4)-G_i(t', \tau', x_1', x_2', x_3', x_4')| \leq g_i\left[|t-t'|^\nu+|\tau-\tau'|^\mu+\sum_{j=1}^{4}|x_j-x_j'|\right].$$

If the real and imaginary parts of $k(t, \tau, u)$ $(x+iy=u)$ belong to class $H_{\mu\nu}$ on a curve L for some μ and ν the author proves that the singular integral equation

$$\varphi(t) = \int_L \frac{k[t, \tau, \varphi(\tau)]}{\tau-t}\, d\tau$$

has a unique solution for $|\lambda|$ sufficiently small. This generalizes the work of W. Pogorzelski [same Ann. **1** (1954), 138–148; MR **16**, 261].

R. C. MacCamy (Pittsburgh, Pa.)

5876:

Sawyer, W. W. On the integral equation

$$kf(x) = \int_1^n (x+y)^{-1}f(y)dy.$$

J. London Math. Soc. **34** (1959), 451–453.

The author shows that the solutions of the integral equation of the title are precisely those solutions of a Fuchsian differential equation of the second order with the four singularities ± 1 and $\pm n$ and with exponents 0, 0 at each of these singularities which are regular both at 1 and at n (this regularity condition being a condition on the accessory parameter in the differential equation). If k is the largest eigenvalue of the integral equation, and $f(x)$ is a corresponding eigenfunction, the author points out that $f(x)/f(-x)$ maps the upper half-plane onto a region consisting of a strip of width π/k from which a circle with diameter k/π has been removed. Since the map must be connected, it follows that no eigenvalue of the integral equation exceeds π, a circumstance which is shown also by Hilbert's inequality.

A. Erdélyi (Pasadena, Calif.)

5877:

Kim, E. I. Conditions for the solvability of a certain class of integro-differential equations. Dokl. Akad. Nauk SSSR **125** (1959), 723–726. (Russian)

Consider an integro-differential equation of the form

$$(*)\quad \psi(y, t)+\sum_{k=1}^{m} A_k\int_0^t d\tau\int_{-\infty}^{\infty} \psi_\eta^{(k)}(\eta, \tau)(t-\tau)^{\frac{1}{2}k-1} \times G(y-\eta, t-\tau)d\eta = \varphi(y, t),$$

where the A_k are constants, $\varphi(y, t)$ is a given function and

$$G(y-\eta, t-\tau) = \frac{1}{2a\sqrt{(\pi(t-\tau))}}\exp\left[\frac{-(y-\eta)^2}{4a^2(t-\tau)}\right]$$

(a is a constant). The author defines the Fourier transform with respect to y of generalized functions and converts (*) into an integral equation in the dual space. Then it is shown that if $\phi(y, t)$ satisfies the inequality $|\phi|\leq C_1 \exp C|y|^{2-\varepsilon}$ $(C>0, \varepsilon>0)$ there exists a unique solution of (*) in a certain class of generalized functions.

C. G. Maple (Ames, Iowa)

FUNCTIONAL ANALYSIS

See also 5573, 5726, 5810, 5812, 5865, 5869, 5950, 6033, 6034, 6035, 6036.

5878:

Ornstein, Donald. Dual vector spaces. Ann. of Math. (2) **69** (1959), 520–534.

Let $\mathscr{D}$ be a division ring, E a left vector space and F a right vector space over $\mathscr{D}$ such that E and F are in duality [see G. W. Mackey, Trans. Amer. Math. Soc. **57** (1945), 155–207; MR **6**, 274]. A subspace is said to be closed if it is equal to its double annihilator. The pair E, F is said to be modular if in either E or F the sum of any two closed subspaces is closed. The pair E, F is called splittable if in either E or F every closed subspace admits a closed complement such that the annihilators span the other space. The author shows that, if E, F is modular, then in one of the spaces all subspaces of countable dimension are closed. A pair of dual spaces E, F is called an $(\aleph_\alpha, \aleph_\beta)$-pair if E has dimension $\aleph_\alpha$ and a basis r_i such that F consists exactly of those functionals which vanish outside a subset of power $<\aleph_\beta$ of the r_i. For $(\aleph_\alpha, \aleph_\beta)$-pairs, modularity and splittability may be connected with properties of $\aleph_\beta$. Let X be a complemented, modular, centerless, complete, atomic lattice. Then X is shown to be either a non-Desarguian projective plane or the lattice of closed subspaces of a modular pair of dual vector spaces.

V. Pták (Prague)

5879:

Marinescu, G. Opérations linéaires dans les espaces vectoriels pseudotopologiques et produits tensoriels. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (**50**) (1958), 49–54.

In earlier notes [cf., for instance, G. Marinescu, Rev. Math. Pures Appl. **2** (1957), 413–417; MR **20** #1189], the author discussed how the study of continuous linear transformations between locally convex topological vector spaces leads to the notion of what he calls pseudotopological structures on vector spaces, which is subsumed by the notion of inductive limits of locally convex topological vector spaces as introduced by N. Bourbaki. In the present note, the author discusses continuous linear transformations between pseudotopological unions and relates them to topological tensor products.

L. Nachbin (Rio de Janeiro)

5880:

Sawashima, Ikuko. On Banach limits in some topological vector spaces. Nat. Sci. Rep. Ochanomizu Univ. **9** (1958), 13–18.

The author generalizes the notion of Banach limits to sequences and functions taking values in a semi-reflexive Hausdorff locally convex space E. For each bounded directed sequence $x=\{x_\alpha\}$, he defines an element $\mathrm{Lim}_\alpha x_\alpha$ in E, which depends linearly on x and belongs to $\overline{\mathrm{Co}}\,\{x_\beta; \beta>\alpha\}$ for all α, where $\overline{\mathrm{Co}}$ denotes the closed convex cover. He then considers bounded functions x from a (totally finite) measure space to E, and proves the existence of a generalized integral $I(x, \sigma)$ for each measurable subset σ, which is linear in x, is completely additive in σ, and belongs to $m(\sigma)\,\overline{\mathrm{Co}}\,x(\sigma)$, where $m(\sigma)$ is the measure of σ.

A. P. Robertson (Lawrence, Kans.)

5881:

Sundaresan, K. Orthogonality in normed linear spaces. J. Indian Math. Soc. (N.S.) **22** (1958), 99–107.

If x is a non-zero element of the normed space B, the angle $\langle x, S\rangle$ between x and the subspace S is defined by $\sin\langle x, S\rangle = d(x/\|x\|, S)$; the latter equals 1 if and only if x is orthogonal to S in the sense used by R. C. James [Duke Math. J. **12** (1945), 291–302; Trans. Amer. Math. Soc. **61** (1947), 265–292; Bull. Amer. Math. Soc. **53** (1947), 559–566; MR **6**, 273; **9**, 42]. If S is the line generated by $y\neq 0$, the angle $\langle x, y\rangle$ between x and y is defined to be $\langle x, S\rangle$. This definition is shown to agree with the usual one if B is an inner product space, and (as shown by James) B must indeed be an inner product space if some of the usual properties of angles are satisfied. The author proves various theorems, based on (and related to) results of James, concerning the existence of elements at a given angle to a given subspace, etc.

R. R. Phelps (Princeton, N.J.)

5882:

Poulsen, Ebbe Thue. Convex sets with dense extreme points. Amer. Math. Monthly **66** (1959), 577–578.

It is shown that in real Hilbert space l^2 there exist closed convex sets (different from a single point) which coincide with the closure of the set of their extreme points. Such is, e.g., the set of all points $(x_1, x_2, \cdots)$ of l^2 satisfying $\sum_{i=1}^{\infty}(2^i x_i)^2 \leq 1$. Similar examples may be constructed in any infinite-dimensional Banach space with a basis. Also, there exist convex sets which are the closure of the set of their vertices [for this notion see, e.g., H. F. Bohnenblust and S. Karlin, Ann. of Math. (2) **62** (1955), 217–229; MR **17**, 177]. The above results were generalized in a recent paper by V. Klee [Math. Ann. **39** (1959), 51–63].

B. Grünbaum (Princeton, N.J.)

5883:

Efimov, N. V.; and Stečkin, S. B. Support properties of sets in Banach spaces and Čebyšev sets. Dokl. Akad. Nauk SSSR **127** (1959), 254–257. (Russian)

If M is a subset of a Banach space E and $a>0$, the a-hull M_a of M is the complement of the union of all the open balls of radius a which miss M. The set M is a-convex if $M = M_a$. The authors sketch a proof that if M is the closure of an open a-convex subset of a uniformly convex Banach space, then there is a set S of second category in the boundary of M such that for each $x \in S$ there is a unique ball of radius a supporting M at x. A subset M is said to be a Čebyšev set if each point in E has a unique nearest point in M. The authors outline a proof (which uses the above result) that in a uniformly convex Banach space with smooth unit ball, every compact Čebyšev set is convex. This generalizes (and is in part based on) their previous results [Dokl. Akad. Nauk SSSR **118** (1958), 17–19; **121** (1958), 582–585; MR **20** #1947, #6026]. An easy corollary is the (known) theorem that the class of Čebyšev subsets of a finite-dimensional space coincides with the class of closed convex sets if and only if the unit ball is smooth and strictly convex. It is pointed out that the version of this theorem given in the first paper cited above is incorrect.

R. R. Phelps (Princeton, N.J.)

5884:

Singer, I. On a theorem of J. D. Weston. J. London Math. Soc. **34** (1959), 320–324.

Let X be a normed linear space, Y a subspace of the conjugate space X^*, and f the canonical mapping of X in Y^*. Then the image of the closed unit ball of X is dense, in the topology $\sigma(Y^*, Y)$, in the closed unit ball of Y^*. Furthermore a necessary and sufficient condition that f be an isometrical isomorphism of X onto Y^* is that the closed unit ball of X be compact in the topology $\sigma(X, Y)$.

P. Civin (Eugene, Ore.)

5885:

Singer, Ivan. Sur quelques théorèmes de W. W. Rogosinski et S. I. Zoukhovitzky. Rev. Math. Pures Appl. **3** (1958), 117–130.

Let E be a Banach space with unit sphere S. Rogosinski [Proc. London Math. Soc. (3) **6** (1956), 175–190; MR **17**, 987] and Zuhovickiĭ [Izv. Akad. Nauk SSSR. Ser. Math. **21** (1957), 409–422; MR **19**, 566] have proved (for the cases $E = L_1(T)$, $T \subset R^n$, and $E = C(Q)$, Q compact metric, respectively) theorems concerning those linear functionals f on E for which there exists an $x \in S$ such that $f(x) = \|f\|$. The author uses a theorem of Choquet [C. R. Acad. Sci. Paris **243** (1956), 699–702; MR **18**, 219] to obtain related results for E a separable Banach space; he then uses these to help re-prove the theorems of Rogosinski and Zuhovickiĭ. {The methods of the latter authors may be extended to obtain analogous theorems for the (not necessarily separable) spaces L_1 (of an arbitrary measure space) and $C(X)$, X compact Hausdorff. See #5887 below.}

R. R. Phelps (Princeton, N.J.)

5886:

Singer, Ivan. A short proof of a theorem of E. Helly. Rev. Math. Pures Appl. **3** (1958), 437–438.

For a similar proof of a more general theorem, see the author's later papers, C. R. Acad. Sci. Paris **247** (1958), 408–411, 846–849 [MR **20** #5423, #5424].

R. R. Phelps (Princeton, N.J.)

5887:

Phelps, R. R. Some subreflexive Banach spaces. Arch. Math. **10** (1959), 162–169.

In this paper the author shows that the following Banach spaces are subreflexive [see R. R. Phelps, Arch. Math. **8** (1957), 444–450; MR **20** #6027]. (i) If S is an arbitrary set, then $c_0(S)$, $l_1(S)$ and $m(S)$ and all their conjugates are subreflexive. (ii) If (S, F, m) is a measure space, then $L_1(S, F, m)$ and all its conjugates are subreflexive. (iii) Every (AL)-space or (AM)-space with a unit and their conjugates are subreflexive. (iv) If S is a Hausdorff topological space, then $C(S)$, the space of all bounded continuous real functions, and each of its conjugates is subreflexive. The same is true for the space $C_0(S)$ of all continuous real functions which vanish at infinity and which are defined on a locally compact Hausdorff space S.

W. A. J. Luxemburg (Pasadena, Calif.)

5888:

Stein, E. M.; and Weiss, Guido. An extension of a theorem of Marcinkiewicz and some of its applications. J. Math. Mech. **8** (1959), 263–284.

Let (M, μ), (N, ν) be two measure spaces, $\mathscr{S}(M)$ the class of simple functions on M, $\mathscr{M}(N)$ the class of measurable functions on N, and let $\lambda(h; y) = \nu\{x \in N;\ |h(x)| > y\}$ where $h \in \mathscr{M}(N)$, $y > 0$. A linear operator T mapping $\mathscr{S}(M)$ into $\mathscr{M}(N)$ is of type (p, q) if there is a constant A

such that $\|Tf\|_q \le A\|f\|_p$ for all $f \in \mathcal{S}(M)$. T is of weak type (p, q) if (*) $\lambda(Tf; y) \le [A\|f\|_p/y]^q$ for any $f \in \mathcal{S}(M)$ and $y > 0$. Let $1 \le p_k \le q_k < \infty$ $(k=0, 1)$, $q_0 \ne q_1$, $0 < t < 1$, $1/p_t = (1-t)/p_0 + t/p_1$, $1/q_t = (1-t)/q_0 + t/q_1$. A theorem of Marcinkiewicz and Zygmund [see Zygmund, J. Math. Pures Appl. **35** (1956), 223–248; MR **18**, 321] asserts that if T is of weak types (p_0, q_0) and (p_1, q_1) then it is of type (p_t, q_t). The authors say that T is of restricted weak type (p, q) if (*) holds for any f of the form $f = f_E$ = characteristic function of a measurable set $E \subset M$. They prove the following theorem: If T is of restricted weak types (p_0, q_0) and (p_1, q_1), then it is of type (p_t, q_t). A similar result holds for restricted types. They show that if T is the Hilbert transform then $\lambda(Tf_E; y) = 2|E|/\sin hy$, and thus obtain a direct proof of the inequalities of M. Riesz. Similar applications are made for the conjugate function, fractional integrals and a theorem of Paley, and various relations between the concepts of "types" and "restricted types" are discussed. *M. Cotlar* (Buenos Aires)

5889:

Bear, H. S. Complex function algebras. Trans. Amer. Math. Soc. **90** (1959), 383–393.

Let X be a compact Hausdorff space and $C(X)$ the algebra of all continuous complex-valued functions on X. Let A be a proper closed subalgebra of $C(X)$ which separates points on X and contains the constants. The author introduces the following notion of the essential set of A: it is the minimal closed subset E of X such that A contains all functions in $C(X)$ which vanish on E. He shows there always is such an essential set. If $E = X$, he calls A an essential algebra on X. Theorem: Assume X is the maximal ideal space of A. Then $A|E$ is a closed proper subalgebra of $C(E)$ whose maximal ideal space is E. (Here $A|E$ denotes the algebra of restrictions of functions in A to E.) Next let A be an essential algebra on X with maximal ideal space X. Then (i) X has no isolated points and (ii) no open-closed subset of X is totally disconnected. The author next considers essential algebras on X which are also maximal, i.e., contained in no larger proper subalgebra of $C(X)$. Helson and Quigley [Proc. Amer. Math. Soc. 8 (1957), 111–114; MR **18**, 911] have studied such algebras. The author now shows: If A is an essential maximal algebra on X then $A|F$ is dense in $C(F)$ for each closed $F \ne X$. Using this fact, he gives a new proof of certain results in the above-mentioned paper, as well as some further results on essential maximal algebras. Finally, he defines an algebra A to be extremal if there exists on its maximal ideal space M no closed algebra A_1 lying properly between A and $C(M)$ and such that A_1 again has as maximal ideal space M. Several properties of essential extremal algebras are established, in particular, such algebras are integral domains.

J. Wermer (Cambridge, Mass.)

5890:

Besov, O. V. On some families of functional spaces. Imbedding and extension theorems. Dokl. Akad. Nauk SSSR **126** (1959), 1163–1165. (Russian)

For a given system of numbers p, θ and $r_1, \cdots, r_n$ such that $1 \le p \le \infty$, $1 \le \theta < \infty$, $r_i > 0$ $(i = 1, 2, \cdots, n)$, the author defines the space $B_{p,\theta}^{(r_1, \cdots, r_n)}$ of all real functions f defined on R_n which have p-integrable generalised (in Sobolev's sense) derivatives $\partial^k f/\partial x_i^k$ $(k = 0, 1, \cdots, \bar{r}_i = [r_i - 1]$; $i = 1, 2, \cdots, n)$ and $\|f\| < \infty$ (where $\|f\|$ is a certain norm too complicated to specify here). Some generalisations of imbedding and extension theorems by Nikolskiĭ concerning the spaces $H_p^{(r_1, \cdots, r_n)}$ [Mat. Sb. (N.S.) **33** (**75**) (1953), 261–326; MR **16**, 453] are stated for the space introduced by the author. *N. Dinculeanu* (Bucarest)

5891:

Gál, István S. The inversion of linear operators acting on distributions. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 79–109.

Let S be a finite-dimensional euclidean space and let D denote the space of Schwartz of indefinitely differentiable functions of compact support on S. Then the author introduces the space of weak distributions on S; this is the algebraic dual of D. The author then proves that the adjoints of certain linear maps L of D into D have inverses in the space of weak distributions; this is the same as proving that L is one-one. Some related results are proven. *L. Ehrenpreis* (New York, N.Y.)

5892:

Łojasiewicz, S. Sur la fixation des variables dans une distribution. Studia Math. **17** (1958), 1–64.

Let E^n denote n-dimensional euclidean space, and let D' be the space of distributions on E^n. For $T \in D'$ and $x_0 \in E^n$, we say that T has a limit at x_0 if the limit $T(x_0' + \lambda x)$ as $\lambda \Rightarrow 0+$ exists for x_0' in a neighborhood of x_0 and is a constant distribution for each x_0' (x is the variable). Contrary to the case $n = 1$, the limit could exist without being constant. (It could depend on the radial direction of approach.) The author gives several necessary and sufficient conditions for the existence of the limit; one such is: There exists a continuous function F and integers such that $T = \partial^p F/\partial x_1^{p_1} \cdots \partial x_n^{p_n}$ with

$$F(x) = (x_1 - x_{01})^{p_1} \cdots (x_n - x_{0n})^{p_n}/p_1! \cdots p_n! + o(|x - x_0|^{p_1 + \cdots + p_n})$$

for x in a neighborhood of x_0. The existence of the limit is equivalent to the existence of $\lim T(x_0' + Ax + s)$ (A a matrix, s a vector) as $|A| + |s| \Rightarrow 0$ with $|A|^m = O(\det A)$, $|s| = O(|A|)$, for x_0' in a neighborhood of x_0.

In case the conditions $|A|^m = O(\det A)$ and $s = O(|A|)$ are relaxed, the author obtains similar types of results. The author shows that a distribution which has values for each x_0 is determined by these values.

L. Ehrenpreis (New York, N.Y.)

5893:

Łojasiewicz, S. Sur le problème de la division. Studia Math. **18** (1959), 87–136.

L. Schwartz conjectured that for f a real analytic function and T a distribution, there exists a distribution S satisfying $fS = T$. In the present paper the author confirms this conjecture. The proof is quite complicated. The idea of the proof is (very roughly) as follows: Let V be the set of real zeros of f. The author obtains a decomposition of V into subvarieties. By means of this decomposition the author proves his main inequality: for x in a sufficiently small neighborhood of a point x_0 in V, we can find a constant d so that $|f(x)| \ge d\rho(x, V)^d$, where $\rho(x, V)$ is the

distance from x to V. In case f is a polynomial the result was obtained by McKibben for two variables and by Hörmander in general. *L. Ehrenpreis* (New York, N.Y.)

5894:

Sato, Mikio. On a generalization of the concept of functions. II. Proc. Japan Acad. **34** (1958), 604–608.

[For part I, see same Proc. **34** (1958), 126–130; MR **20** #2618.] Let X be a topological space; $L(X)$ is the totality of open sets on X, and F is a sheaf on X. Let S be a closed set in X; for any $D \in L(X)$ we define $G^0(S, D, F)$ and $G^1(S, D, F)$ as the kernel and cokernel of the natural homomorphism $H^0(D, F) \to H^0(D-S, F)$ respectively. For $n \geq 2$ we set $G^n(S, D, F) = H^{n-1}(D-S, F)$. If $D > D'$ we have natural homomorphisms of $G^n(S, D, F) \to G^n(S, D', F)$, so for each n the $G^n(S, D, F)$ constitute a pre-sheaf; we denote by $\mathrm{Dist}^n(S, X, F)$ the corresponding sheaf. The elements of $H^0(S, \mathrm{Dist}^n(S, X, F))$ are called F distributions of degree n over S.

The author also considers the case where X is a complex manifold and F is a locally free sheaf.

L. Ehrenpreis (New York, N.Y.)

5895:

Fišman, K. M. Linear mappings of analytic spaces. Dokl. Akad. Nauk SSSR **127** (1959), 40–43. (Russian)

The author formulates the following general theorem (in a different notation): Let $A(r)$ for $r > 0$ be a nest of Banach spaces such that $A(r)$ is everywhere dense in $A(s)$ for $s < r$, and the norm is a non-decreasing function of r. Let $P(r)$ be the projective limit of $A(s)$ for $s < r$, and $Q(r)$ the inductive limit for $s > r$. Let $B(u)$ be another similar nest with projective and inductive limits $X(u)$ and $Y(u)$. Then a linear operator H is a continuous mapping of (1) $P(r)$ to $X(u)$, (2) $Q(r)$ to $Y(u)$, (3) $P(r)$ to $Y(u)$, (4) $Q(r)$ to $X(u)$, if and only if there is a function $v(s)$ such that H is a bounded linear operator from $A(s)$ to $B(v(s))$ and (1) $v(s)$ is defined in an interval with upper bound r, $v(s) < u$, and the lim sup of $v(s)$ as s goes to r in the interval is u, (2) $v(s)$ is defined for $s > r$ and $v(s) > u$, (3) $v(s)$ is defined for some $s < r$ and $v(s) > u$, (4) $v(s)$ is defined for $s > r$ and $v(s) < u$. There are at least four misprints in the published version of this theorem. The author shows how this result can be used to generalize and unify a number of known results about mappings involving analytic functions.

D. C. Kleinecke (Berkeley, Calif.)

5896:

Krabbe, Gregers L. Normal operators on the Banach space $L^p(-\infty, \infty)$. I. Bounded operators. Bull. Amer. Math. Soc. **65** (1959), 270–272.

Let us denote by: $\mathscr{B}$, the Boolean ring generated by the semi-closed intervals of the plane; $\mathscr{D}$, the class of all bounded complex-valued functions defined on the real line R, whose real and imaginary parts are piecewise monotone; $\mathscr{T}_p$, the algebra of all linear bounded operators on $L^p = L^p(R)$; L^+, the intersection of the family $\{L^p; 1 < p < \infty\}$; $\mathscr{T}$, the set of all operators T on L^+ such that $\|T\|_q < \infty$ whenever $1 < q < \infty$; T_p, the unique extension of T to L^p, where $T \in \mathscr{T}$ and $1 < p < \infty$. The following theorem is stated: If $T \in \mathscr{T}$ and if there exists $f \in \mathscr{D}$ such that (Fourier transform of Tx) = (Fourier transform of x)$\cdot f$ for all $x \in L^+$, then for any $1 < p < \infty$ there exists a homomorphic mapping E_p of the ring $\mathscr{B}$ into the algebra $\mathscr{T}_p$ such that $T_p = \int \lambda E_p(d\lambda)$. The integral is in the Riemann-Stieltjes sense, in the strong operator-topology when $p \geq 2$ and in the weak topology when $1 < p < \infty$.

N. Dinculeanu (Bucharest)

5897:

Orlicz, W. On the generation of l^α-space and L^α-space. Acta Math. Sinica **9** (1959), 150–155. (Chinese)

In this article, the author discusses some of the generalizations of l^α and L^α spaces, which he introduced in 1932, and reviews some of the work that has since been done in this direction. Most of the results mentioned were reported in a recent paper by S. Mazur and W. Orlicz [Studia Math. **17** (1958), 97–119; MR **20** #4780].

Choy-tak Taam (Washington, D.C.)

5898:

Milnes, Harold Willis. Convexity of Orlicz spaces. Pacific J. Math. **7** (1957), 1451–1483.

Let $v = \varphi(u)$ be a non-zero monotonically nondecreasing left-continuous function defined for $u \geq 0$ with $\varphi(0) = 0$. Let $u = \psi(v)$ be the function inverse to $\varphi(u)$ which is defined by the relations: $\psi(0) = 0$; $\psi(v) = u$ if $\varphi(u) = v$ and u is a point of continuity for $\varphi(u)$; $\psi(v) = \psi(v-)$; if $\varphi(u) \neq \varphi(u+)$, then $\psi(v) = u$ for all v such that $\varphi(u) \leq v \leq \varphi(u+)$; if $\lim_{u\to\infty} \varphi(u) = p < +\infty$, then $\psi(v) = +\infty$ for all $v \geq p$. We define $\Phi(u) = \int_0^u \varphi(t)dt$ and $\Psi(v) = \int_0^v \psi(t)dt$. If (X, Σ) is a measure space with a σ-finite measure μ, the Orlicz space L_Φ is defined to be the collection of all μ-measurable functions f, such that $\|f\|_\Phi = \sup \int_X |f(x)|g(x)d\mu < +\infty$, where the supremum is taken for all $g(x) \geq 0$ such that $\int_X \Psi(g)d\mu \leq 1$. L_Φ is a Banach space with the norm $\|f\|_\Phi$.

The author proves that a necessary and sufficient condition for L_Φ to be strictly convex is that $\psi(v)$ and $\Psi(v)$ should be continuous in the extended sense (range is in the extended reals). He also proves that if $\mu(X) = +\infty$ then L_Φ is uniformly convex if and only if (i) ψ and Ψ are continuous in the extended sense, (ii) there is a constant $0 < N < +\infty$ such that if $u > 0$, $\Phi(2u)/\Phi(u) \leq N$, and (iii) for each constant $0 < \varepsilon < \frac{1}{2}$ there is a constant $1 < R_\varepsilon < +\infty$ such that if $u > 0$, $\varphi(u)/\varphi((1-\varepsilon)u) > R_\varepsilon$. If $\mu(X) < +\infty$ then L_Φ is uniformly convex if and only if $0 < \lim \sup_{u\to\infty} \Phi(2u)/\Phi(u) < +\infty$ and for each ε such that $0 < \varepsilon < \frac{1}{2}$, $\lim \inf_{u\to\infty} \varphi(u)/\varphi((1-\varepsilon)u) > 1$.

L. Brown (Detroit, Mich.)

5899:

Štraus, A. V. Characteristic functions of linear operators. Dokl. Akad. Nauk SSSR **126** (1959), 514–516. (Russian)

The main aim of this note is to extend the theory of the characteristic function of a linear operator [M. S. Brodskiĭ and M. S. Livšic, Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1 (79), 3–85; MR **20** #7221] to a much wider class of operators.

Let H be a Hilbert space and let A be a linear operator with dense domain D_A and having a non-empty resolvent set Λ_A; then the resolvent set Λ_{A^*} of A^* is also non-empty. A vector space L with a non-degenerate but possibly indefinite inner product $[\varphi, \psi]$ is called a boundary space of A if there is a linear mapping Γ of D_A onto L such that

$$[\Gamma f, \Gamma g] = \frac{1}{i}[(Af, g) - (f, Ag)] \quad (f, g \in D_A).$$

The mapping Γ is the corresponding boundary operator of A.

The space L and the mapping Γ can be constructed as follows: let

$$G_A = \{f : f \in D_A \cap D_{A^*}, (A-A^*)f = 0\},$$

take $L = D_A/G_A$, and define the inner product in L by the equation

$$[\xi, \eta] = \frac{1}{i}[(Af, g)-(f, Ag)] \quad (f \in \xi, g \in \eta);$$

the mapping Γ is then the natural mapping of D_A onto L.

A boundary space L' and a boundary operator Γ' are defined similarly for A^*, except for the sign of the inner product:

$$[\Gamma'f', \Gamma'g']' = \frac{1}{i}[(f', A^*g')-(A^*f', g')] \quad (f', g' \in D_{A^*}).$$

For each $\lambda \in \Lambda_{A^*}$ we define the linear operator

$$S_\lambda = (A^*-\lambda E)^{-1}(A-\lambda E),$$

which takes D_A into D_{A^*} [E is the identity operator]. Finally we can define a linear operator $X(\lambda)$ taking L into L' by the equation

$$X(\lambda)\Gamma f = \Gamma' S_\lambda f \quad (f \in D_A, \lambda \in \Lambda_{A^*}).$$

Then $X(\lambda)$ is called the characteristic function of A.

For each $\lambda \in \Lambda_{A^*}$, let

$$S_\lambda' = (A-\bar{\lambda}E)^{-1}(A^*-\bar{\lambda}E),$$

which maps D_{A^*} into D_A.

The author now states the following results without proof. Let H_1 be the closed vector subspace of H spanned by

$$\{f-S_\lambda f, f'-S_\lambda' f' : f \in D_A, f' \in D_{A^*}, \lambda \in \Lambda_{A^*}\};$$

then H_1 reduces A; on H_1, A is determined, up to unitary equivalence, by its characteristic function $X(\lambda)$; on the orthogonal complement H_2 of H_1, the operator A is self-adjoint.

Furthermore, suppose that $A^{(1)}$ and $A^{(2)}$ are simple linear operators in the Hilbert spaces $H^{(1)}$ and $H^{(2)}$ respectively, both having a dense domain and a non-empty resolvent set (a simple operator is one that is not self-adjoint on any non-zero subspace that reduces it); then $A^{(1)}$ and $A^{(2)}$ are unitarily equivalent if and only if their characteristic functions coincide. {Reviewer's comment: The meaning of the last condition is not fully explained.}

Several other relations between various operators are also given. *F. Smithies* (Cambridge, England)

5900:

Štraus, A. V. A multiplication theorem for characteristic functions of linear operators. Dokl. Akad. Nauk SSSR **126** (1959), 723–726. (Russian)

This note falls into two parts. In the first part the author generalizes the notion of a linkage of two linear operators, defined in two orthogonal subspaces of a Hilbert space H [M. S. Brodskiĭ and M. S. Livšic, Uspehi Mat. Nauk (N.S.) **13** (1958), no. 1 (79), 3–85; MR **20** #7221; p. 11] to cases where the operators are unbounded. When certain additional conditions are satisfied, the linkage is said to be regular. With the definition of characteristic function given in his previous note [review above], the author states the following result: if A is a regular linkage of A_1 and A_2, and the resolvent sets of A_1^* and A_2^* have a non-void intersection Λ, then the characteristic functions of the three operators satisfy the relation

$$X(\lambda) = X_2(\lambda)X_1(\lambda) \quad (\lambda \in \Lambda).$$

In the second part of the note the author indicates that an arbitrary operator A in a Hilbert space H can be represented as one constituent of a linkage $\tilde{A}$ in an extension $\tilde{H}$ of H in such a way that the characteristic function $X(\lambda)$ of $\tilde{A}$ (called the generalized characteristic function of A) determines A up to unitary equivalence.

F. Smithies (Cambridge, England)

5901:

Nelson, Edward. Analytic vectors. Ann. of Math. (2) **70** (1959), 572–615.

Let A be an unbounded operator in a Banach space. A vector x is said to be analytic for A if $\sum (\|A^n x\|/n!)s^n$ converges for small s ($\|Bx\| = \infty$ if x is not in the domain of B). Sufficient conditions are given in terms of higher commutators $(\mathrm{Ad}\, B)^n A$ in order that the analytic vectors for A should include the analytic vectors for B. This is applied to the study of first order differential operators on a manifold, to the analysis of the relation between unitary representations of a Lie group and representations of its Lie algebra, and to the study of elliptic partial differential operators, wherein it is demonstrated that an analytic vector x for an elliptic operator A with analytic coefficients is an analytic function. Analytic vectors for unitary representations of Lie groups are also constructed by smoothing with kernels derived from the heat equation.

J. T. Schwartz (Berkeley, Calif.)

5902:

Itô, Takasi. On the commutative family of subnormal operators. J. Fac. Sci. Hokkaido Univ. Ser. I **14** (1958), 1–15.

Halmos [Summa Brasil. Math. **2** (1950), 125–134; MR **13**, 359] proposed calling an operator A on the Hilbert space H subnormal if there exists a larger space K containing H as a subspace and a normal N on K such that $NH \subset H$ and $N|H = A$. The obviously necessary positivity condition $\sum_{m,n} (A^m x_n, A^n x_m) \geq 0$ ($\{x_0, x_1, \cdots\}$ an arbitrary finite sequence of vectors in H) was shown by Bram [Duke Math. J. **22** (1955), 75–94; MR **16**, 835] to be sufficient for A to be subnormal. Observing that the problem of extending A is the same as the problem of extending the semigroup $\{1, A, A^2, \cdots\}$ the author obtains the following generalization. Let Γ denote any additive abelian semigroup with 0; A_γ ($\gamma \in \Gamma$, H) is a representation of Γ on H if each A_γ is a bounded operator on H with $A_0 = 1$ and $A_{\gamma+\delta} = A_\gamma A_\delta$. Call A_γ ($\gamma \in \Gamma$, H) positive definite if $\sum_{\gamma,\delta} (A_\gamma x_\delta, A_\delta x_\gamma) \geq 0$ for all families $\{x_\gamma\}_{\gamma\in\Gamma}$ of vectors in H with all but a finite number of $x_\gamma = 0$. Then (Th. 1) a necessary and sufficient condition that A_γ ($\gamma \in \Gamma$, H) possess a normal extension N_γ ($\gamma \in \Gamma$, K) (each N_γ normal on $K \supset H$ and, for each γ, $N_\gamma H \subset H$, $N_\gamma|H = A_\gamma$) is that A_γ ($\gamma \in \Gamma$, H) be positive definite. The construction and proof parallel those of Halmos and Bram [loc. cit.]; the extension is minimal in the obvious sense, and the minimal extension is unique up to isomorphism.

Following Bram, the author also considers the problem of extending to K an operator B that commutes with all

A_γ, and obtains the appropriate generalization. New proofs are given of Bram's results on the spectrum of a subnormal operator A and on the weakly closed (non-selfadjoint) ring it generates, systematic use of the ideas of semigroup representation and positive definiteness permitting some simplifications of the proofs. Other noteworthy results: (1) A is subnormal on H if and only if $\exp(tA)$ $(t \geqq 0, H)$ is positive definite; (2) a weakly continuous one-parameter semigroup A_t $(t \geqq 0)$ is positive definite (self representation) if and only if each A_t is subnormal, and in this case the minimal normal extension N_t is also weakly continuous; (3) an absolutely arbitrary representation V_γ $(\gamma \in \Gamma, H)$ of an abelian semigroup by isometries V_γ admits a unitary extension. This last provides a broad generalization of a result due to Cooper [Ann. of Math. (2) **48** (1947), 827–842; MR **10**, 257]. *A. Brown* (Houston, Tex.)

5903:

Miyadera, Isao. A note on contraction semi-groups of operators. Tôhoku Math. J. (2) **11** (1959), 98–105.

Let $\Sigma = [T(s); s \geqq 0]$ denote a strongly continuous semi-group of positive contraction operators on an abstract (L)-space X. A second such semi-group $\Sigma' = [T'(s); s \geqq 0]$ is said to dominate Σ if $T'(s)x \geqq T(s)x$ for all $x \geqq 0$ and $s > 0$. This paper is concerned with the generation of dominating semi-groups of this kind, a problem previously investigated by G. E. H. Reuter [Math. Scand. **3** (1955), 275–280; MR **17**, 988]. Defining the positive linear functional $(e, x) = |x^+| - |x^-|$ on X, the author proves: Suppose A generates a semi-group of positive contraction operators Σ and let B be a linear operator with domain $D(B) > D(A)$. Then $A + B$ generates a positive contraction [transition] semi-group Σ' dominating Σ if and only if $Bx \geqq 0$ for all $x \geqq 0$ in $D(A)$, $(e, Bx) \leqq -(e, Ax)$ $[= -(e, Ax)]$ for all $x \geqq 0$ in $D(A)$, and either (a) $(I - BR(\lambda; A))[X] = X$ for each $\lambda > 0$, or (b) $\sum_0^\infty |BR(\lambda; A)^n y| < \infty$ for each $\lambda > 0$ and $y \geqq 0$. If, in addition, $Bx = 0$ for each x in $[x; x \in D(A), (e, Ax) = 0]$, then there exists a $c_B \geqq 0$ in X with $|c_B| \leqq 1$ such that $Bx = -(e, Ax)c_B$ for all x in $D(A)$. As Reuter has shown, the semi-group Σ' will consist of transition operators if and only if $|c_B| = 1$. The author also shows by example that there may exist contraction semi-groups Σ' dominating Σ other than those described above.

R. S. Phillips (Los Angeles, Calif.)

5904:

Balakrishnan, A. V. An operational calculus for infinitesimal generators of semigroups. Trans. Amer. Math. Soc. **91** (1959), 330–353.

The author develops an operational calculus for infinitesimal generators of semi-groups of operators defined on a Banach space, B, along the lines developed by Hille and Phillips [*Functional analysis and semi-groups*, Amer. Math. Soc., Providence, R.I., 1957; MR **19**, 664]. The author's calculus includes the Hille-Phillips calculus as well as operators which are unbounded.

Let $\{T(\xi); \xi > 0\}$ be a strongly continuous semi-group on B with $w = \inf\{\xi^{-1} \log \|T(\xi)\|; \xi > 0\} > -\infty$, and $S(w)$ the collection of all countably additive set functions α defined on the Borel sets of $[0, \infty)$ such that $\int \|T(\xi)\| d|\alpha| < \infty$. If $\varphi(\lambda) = \Phi(\lambda; \alpha)$, $\lambda \leqq w$, is the Laplace-Stieltjes transform of $d\alpha$ and A the infinitesimal generator of $T(\xi)$, then the Hille-Phillips calculus consists of the operators

$$\theta(\alpha)x = \varphi(A)x = \int_0^\infty T(\xi)x\,d\alpha(\xi), \quad x \in B.$$

All of the operators $\varphi(A)$ are bounded.

Let $L(w)$ be the subset of absolutely continuous measures in $S(w)$. The author considers "multiplier transforms", C_w, with domain and range in $L(w)$. More specifically, let $\psi(\lambda)$ be defined for $\lambda \leqq w$ and Lebesgue measurable. The element $f \in L(w)$ is in the domain of C_w with $C_w f = g$ if and only if $\psi(\lambda)\Phi(\lambda; f) = \Phi(\lambda; g)$. Only those transforms C_w are considered which have domain dense in $L(w)$. Let C_0 be the operator whose domain consists of all elements of the form $\theta(f)x$, $f \in L(w)$, $x \in B$, and defined by $C_0\theta(f)x = \theta(C_w f)x$. It is easily shown that C_0 can be extended to a closed linear operator, C, with domain dense in B. The operator $\psi(A)$ is defined as C. The author also studies the spectral properties of $\psi(A)$.

As an application of the calculus developed in this paper the author defines non-integral powers of infinitesimal generators of a special class of semi-groups and a connection is shown between these operators and Riemann-Liouville fractional integrals. The author also studies semi-groups of the form $S(t)x = \int T(\xi)x\,d\alpha_t(\xi)$, where $\{\alpha_t\}$ forms a semi-group under convolution.

A. Devinatz (St. Louis, Mo.)

5905:

Naĭmark, M. A. The decomposition into irreducible representations of the tensor product of a representation of the principal series and a representation of the complementary series of the proper Lorentz group. Dokl. Akad. Nauk SSSR **125** (1959), 1196–1199. (Russian)

Supplementing a previous paper [Dokl. Akad. Nauk SSSR **119** (1958), 872–875; MR **20** #7228], the author obtains the reduction, into a continuous sum of irreducible constituents, of the tensor product of a member of the principal series and a member of the complementary series of irreducible representations of the proper Lorentz group. The irreducible constituents all belong to the principal series. *A. J. Coleman* (Kingston, Ont.)

5906:

★Neumark, M. A. Normierte Algebren. Hochschulbücher für Mathematik, Bd. 45. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959. 572 pp. DM 48.

Translation, essentially unchanged except for some corrections and some additions to the bibliography, from *Normirovannye kol'ca*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956 [MR **19**, 870].

5907:

Kaplansky, I. Topological algebra. Notas Mat. No. 16 (1959), v + 87 pp.

These notes were drawn up in 1952 and were issued in their present form with no amendment in 1959. The author managed to intercalate a second preface indicating the outdatedness of the presentation. It is the reviewer's opinion that no apologies are required. These notes constitute a whirlwind tour of many interesting domains of topological algebra. Proofs are rarely more than sketched and are, more often than not, omitted. A discursive section (16, "References and remarks") provides the interested reader with historical data and excel-

lent bibliographical material where proofs left incomplete in the notes can be found in detail.

Although a present-day student of the field might first be led to read Loomis [*An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953; MR **14**, 883], Naĭmark [*Normirovannye kol'ca*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 870], or Kaplansky's [*Some aspects of analysis and probability*, pp. 1–34, Wiley, New York, 1958; MR **21** #286] and Hewitt's [ibid., pp. 105–168; MR **21** #2159] articles, he will ultimately profit from the notes under review. The tendency to emphasize the important and to gloss over the nagging details or the less interesting by-product theorems is rather pronounced, albeit instructive. These notes are almost a course in mathematical taste—conforming to the author's prejudices. To flesh out the skeleton with proofs and clarifications is a major task. To accept the presentation as a perceptive outline of the field is easier and much more to the point.

In 87 broadly spaced, typewritten pages the following topics are touched: (1) Rings; (2) Normed linear spaces; (3) The spectrum; (4) Ideals in function algebras; (5) The group algebra of a discrete abelian group; (6) Topological divisors of 0; (7) *-Algebras; (8) Continuity theorems; (9) Integrating to get a square root; (10) Introduction to Hilbert space; (11) States and representations; (12) *-Algebras, continued; (13) Weakly closed algebras; (14) Algebras of type I; (15) Spatial theory of algebras of type I.

A bibliography, current in 1952, is offered together with an appendix giving Fukamiya's result on B^* and C^* algebras. We note that unreviewed at this writing is Ono's strengthening of Fukamiya's theorem [J. Math. Soc. Japan **11** (1959), 146–158]: A Banach algebra with involution satisfying the usual conjugate linear, etc., requirements, and only $\|x^*x\| = \|x^*\|\,\|x\|$, is a C^*-algebra.

B. R. Gelbaum (Minneapolis, Minn.)

5908:

Kasahara, Shouro. Representation of some topological algebras. II. Proc. Japan Acad. **35** (1959), 89–94.

The paper under review is a continuation of part I [same Proc. **34** (1958), 355–360; MR **20** #6485] in which it was proved that a topological algebra E satisfying certain conditions is isomorphic to a subalgebra, which includes all continuous linear maps of finite rank, of $L(X, X)$ for some locally convex space X. Here $L(X, X)$ stands for the space of all continuous linear maps of X into itself with the topology depending on the conditions E satisfies. In the present paper it is established that such X is unique up to topological isomorphism. Before arriving at this result, various properties of elements of rank 1 (p is of rank 1 if, for each x, $pxp = \lambda p$ for some scalar λ) are proved.

I. Namioka (Ithaca, N.Y.)

5909:

Mackey, George W. Commutative Banach algebras. Notas Mat. No. 17 (1959), 210 pp.

In this set of notes are treated classification of commutative Banach algebras, ideal theory for Banach algebras, and classification of representations of commutative Banach algebras. An attempt is made to indicate the main sources in the literature of the material presented, although no claim to completeness is made. The course from which these notes are taken was given at Harvard University in the spring of 1951. *From the introduction*

5910:

Johnson, G. Philip. Spaces of functions with values in a Banach algebra. Trans. Amer. Math. Soc. **92** (1959), 411–429.

Let A be a commutative Banach algebra with the norm $\|\ \|_A$, and G be a locally compact abelian group with the Haar measure dx. Denote by $B^1(=B^1(G, A))$ the set of all Bochner integrable functions from G to A. B^1 becomes a commutative Banach algebra by the multiplication of the convolution formula relative to the Bochner integral and by the norm $\|f\|_B = \int_G \|f(x)\|_A dx$. The author discusses for B^1 the Gelfand representation and proves an isomorphism theorem of Wendel type. Let $\mathcal{M}_A$ and $\mathcal{M}_B$ be the spaces of regular maximal ideals of A and B with the weak topologies, respectively, and let $\hat{G}$ be the character group of G. The first theorem is the following: $\hat{G} \times \mathcal{M}_A$ and $\mathcal{M}_B$ are homeomorphic by a mapping which is defined by a generalized integral formula for the Fourier transform. The second one reads: when A is a commutative group algebra $L'(H)$, $B^1 = L'(G \times H)$ always holds. In the final part, an isomorphism theorem between the algebras of the forms B^1 for two pairs (G, A) and $(\tilde{G}, \tilde{A})$ is given. It reads: Let τ [resp. $\mathfrak{T}$] be an algebraic and topological isomorphism of G onto $\tilde{G}$[resp. A onto $\tilde{A}$], and let $\alpha \in \hat{G}$. Let $k > 0$ be the constant number determined by $dx = kd\tau x$. Then the triple $(\mathfrak{T}, \tau, \alpha)$ determines uniquely an isomorphism $T = T(\mathfrak{T}, \tau, \alpha)$ of $B^1(G, A)$ onto $B^1(\tilde{G}, \tilde{A})$ by $(Tf)(\tau x) = k\alpha(x)\mathfrak{T}[f(x)]$, and T is isometric if and only if $\mathfrak{T}$ is so. The converse is stated in the following way. Let T be an isometric isomorphism of $B^1(G, A)$ onto $B^1(\tilde{G}, \tilde{A})$. Then, by the first theorem, T induces a correspondence $(\alpha, M) \to (\tilde{\alpha}_{\alpha,M}, \tilde{M}_{\alpha,M})$ between $\hat{G} \times \mathcal{M}_A$ and $\hat{\tilde{G}} \times \mathcal{M}_{\tilde{A}}$. If this correspondence satisfies both the conditions that (1) for every $\alpha_1 \in \hat{G}$ there exists $M_1 \in \mathcal{M}_A$ such that $T[\bigcap_{\mathfrak{M}_A} (\alpha_1, M)] = \bigcap_{\mathfrak{M}_{\tilde{A}}} (\tilde{\alpha}_{\alpha_1,M_1}, \tilde{M})$, and (2) for every $M_2 \in \mathcal{M}_A$ there exists $\alpha_2 \in \hat{G}$ such that $T[\bigcap_{\hat{G}} (\alpha, M_2)] = \bigcap_{\hat{\tilde{G}}} (\tilde{\alpha}, \tilde{M}_{\alpha_2,M_2})$, and if A is semi-simple, then T becomes precisely the form $T(\mathfrak{T}, \tau, \alpha)$ for some $\mathfrak{T}, \tau, \alpha$. {The notion of B^1 is also discussed by A. Hausner, who obtains results similar to those (especially the first theorem) in this paper [cf. Pacific. J. Math. **7** (1957), 1603–1610; MR **20** #1931], and describes some theorems in harmonic analysis [cf. Proc. Amer. Math. Soc. **10** (1959), 1–10; MR **21** #3777]. While using the notion of tensor (= direct, cross) product between two Banach algebras, B. R. Gelbaum discusses more general cases and applies them to the algebra B^1 [cf. Canad. J. Math. **11** (1959), 297–310; MR **21** #2922].}

H. Umegaki (Tokyo)

5911:

McCarthy, Charles A. On open mappings in Banach algebras. J. Math. Mech. **8** (1959), 415–418.

$\mathfrak{A}$ is a complex Banach algebra with unit. If f is a function holomorphic on an open set U of the complex plane, and if the spectrum $\sigma(a)$ of $a \in \mathfrak{A}$ lies in U, then $f(a)$ is defined, $f(a) \in \mathfrak{A}$. Let $f(\mathfrak{A}) = \{f(b) : b \in \mathfrak{A}, \sigma(b) \subset U\}$. The author considers the problem: When is $f(a)$ an inner point of $f(\mathfrak{A})$? He proves four theorems, including an application to the case $f(z) = \exp(z)$. Two results are: If f is one-to-one on U then there is a neighborhood of

$0 \in \mathfrak{A}$ and a continuous map $x \to y(x) \in \mathfrak{A}$ defined in this neighborhood such that $\sigma(a+y(x)) \subset U$ and $f(a+y(x)) = f(a)+x$. Next, if f is holomorphic on U and one-to-one on $V \subset U$; if $f(\sigma(a)) \subset f(V)$; then there is a $b \in \mathfrak{A}$ such that $\sigma(b) \subset V$ and $f(b) = f(a)$. Thus by the first result, $f(a)$ is an interior point of $f(\mathfrak{A})$. *E. R. Lorch* (New York, N.Y.)

5912:

McCarthy, Charles A. On open mappings in Banach algebras. II. Bull. Amer. Math. Soc. **65** (1959), 66.

Proof in three paragraphs of two facts concerning expressions of the type $f(a)$ where a belongs to a complex Banach algebra and f is holomorphic in a region including the spectrum of a. *E. R. Lorch* (New York, N.Y.)

5913:

Olubummo, A. $B^\sharp$-algebras with a certain set of left completely continuous elements. J. London Math. Soc. **34** (1959), 367–369.

Continuing his study of $B^\sharp$-algebras [same J. **32** (1954), 270–276; MR **19**, 665] the author proves that a $B^\sharp$-algebra in which every element with no right reverse is left-completely-continuous, is isometrically isomorphic to the $B(\infty)$-sum of its finite-dimensional two-sided ideals.

R. Arens (Los Angeles, Calif.)

5914:

Rajagopalan, M. Classification of algebras. J. Indian Math. Soc. (N.S.) **22** (1958), 109–116.

Let H be a Hilbert space and also a $*$-algebra over the complex numbers which (a) has a unit element, and (b) satisfies the postulates for an H^*-algebra [see Loomis, *Abstract harmonic analysis*, Van Nostrand, Toronto-New York-London, 1953; MR **14**, 883; p. 100] except that, instead of $\|xy\| \leq \|x\| \|y\|$, we assume only that left and right multiplication by elements of H are continuous operations. The author shows that under these conditions we have $\|xy\| \leq K\|x\| \|y\|$ for some positive K; and hence that H is isomorphic to a finite direct sum of $n \times n$ matrix algebras. *J. M. G. Fell* (Seattle, Wash.)

5915:

Sakai, Shôichirô. On linear functionals of W^*-algebras. Proc. Japan Acad. **34** (1958), 571–574.

This paper gives a polar decomposition of linear functionals on a W^*-algebra M. In particular, an arbitrary σ-weakly continuous linear functional on M can be uniquely factored into a composite of a partial isometry in M and a positive normal functional.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

5916:

Ono, Tamio. Local theory of rings of operators. I. J. Math. Soc. Japan **10** (1958), 184–216.

This paper is a development of the theory of AW^*-algebras [I. Kaplansky, Ann. of Math. (2) **53** (1951), 235–249; MR **13**, 48] utilizing the author's concept of "local" property (too complicated to be given here). He discusses dimension and trace functions, concluding with the following theorems. 1. Let R be a finite AW^*-algebra with a complete set of states $\{f\}$ of R such that for any orthogonal system of projections of R, the vanishing of the function f on each projection yields its vanishing on the sum; then R has a trace. 2. Let R be a finite AW^*-algebra and C a commutative AW^*-algebra contained in the center R_0 of R. Further we assume that R has a complete set of C-valued states $\{f\}$ such that f is completely additive on the projections of R_0 and that, for any orthogonal system of projections of R, the vanishing of f on each projection yields its vanishing on the sum. Then R has a trace. These theorems generalize results of Yen [Duke Math. J. **23** (1956), 207–221; MR **16**, 1033] and Goldman [ibid. **23** (1956), 23–34; MR **17**, 512] respectively.

E. L. Griffin, Jr. (Ann Arbor, Mich.)

5917:

Shields, Paul C. A new topology for von Neumann algebras. Bull. Amer. Math. Soc. **65** (1959), 267–269.

Four topologies exist in a von Neumann algebra $\mathfrak{A}$ which are weaker than the uniform topology. For these topologies, it is known that at least one of the following properties fails: (1) The maps $a \to ab$ and $a \to ba$ are continuous, (2) $(a, b) \to ab$ is continuous for $\|a\| \leq 1$, (3) $a \to a^*$ is continuous. Using theorems of Dixmier [Bull. Soc. Math. France **81** (1953), 9–39; MR **15**, 539] and S. Sakai [Pacific J. Math. **6** (1956), 763–773; MR **18**, 811], considering $\mathfrak{A}$ as the adjoint of a Banach space X, the author introduces in $\mathfrak{A}$ a new topology, satisfying (1)–(3), by taking as a basis of neighborhoods of zero the sets $V_0(x_1, \cdots, x_n) = \{a \mid \|R_a x_i\| \leq 1$ and $\|L_a x_i\| \leq 1$, for $i = 1, 2, \cdots, n\}$, where $R_a x(b) = x(ba)$ and $L_a x(b) = x(ab)$ for $x \in X$. Moreover, the author announces that the topology yields the same continuous linear functionals as the ultra-weak topology. As a consequence, the author points out that the Kaplansky density theorem [ibid. **1** (1951), 227–232; MR **14**, 291] has a simple proof. *M. Nakamura* (Asiya)

5918:

Gil de Lamadrid, Jesús. Topology of mappings and differentiation processes. Illinois J. Math. **3** (1959), 408–420.

Let E, F be locally convex topological real vector spaces, f a mapping of $H \subset E$ into F. For $h \in H$, $x \in E$, $\lambda \in R$ (the reals), $\lambda \neq 0$, put $\delta_\lambda f(h)(x) = [f(h+\lambda x) - f(h)]/\lambda$ if $h + \lambda x \in H$, $\delta_\lambda f(h)(x) = 0$ otherwise. Let τ be a topology of the set $\mathscr{F}(E, F)$ of all (not necessarily continuous or linear) mappings of E into F (as a rule, τ is the topology of uniform convergence over an appropriate family of subsets of E). If, for $\lambda \to 0$, $\delta_\lambda f(h)$ converges to an element $f'(h)$ of $\mathscr{F}(E, F)$, then $f'(h)$ is called the derivative of f at h. If $f'(h)$ exists at every $h \in H$, then the mapping $(h, x) \to f'(h)(x)$ is called the "variation" df of f. Higher derivatives (variations, differentials) are mentioned briefly. In connection with the author's approach, various properties of $\mathscr{F}(H, F)$, $H \subset E$, under suitable topologies, are considered. It is shown that Gâteaux and Fréchet differentiations appear as special cases of the above definition (if τ is the point-open, or, respectively, bounded-open topology). Some theorems of E. H. Rothe [Duke Math. J. **15** (1948), 421–431; MR **10**, 548], concerning properties (for $F = R$) of f implied by the compactness of the derivative, are generalized. There is a result, for the general Gâteaux case, corresponding to the formula $f(b) - f(a) = \int_a^b f'(x)dx$, and various other theorems are given. *M. Katětov* (Prague)

5919:

Nicolescu, Lilly Jeanne. On direct second order differentiability of the second order in Fréchet's or Gâteaux's sense. Rev. Math. Pures Appl. **3** (1958), 217–223.

Let X and Y be normed linear spaces and let $f: X \to Y$. Define

$$\Delta'\Delta f(x) = f(x+h+h') - f(x+h) - f(x+h') + f(x).$$

If there exists a continuous bilinear transformation $\phi: X \times X \to Y$ such that

$$\|h\|^{-1}\|h'\|^{-1}\|\Delta'\Delta f(x) - \phi(x; h, h')\| \to 0$$

as $\|h\| \to 0$ and $\|h'\| \to 0$, then f is said to be directly second-order differentiable in Fréchet's sense at x and ϕ is the Fréchet direct second-order differential. The Gâteaux direct second-order derivative is defined as

$$\lim_{t, t' \to 0} (tt')^{-1}[f(x+th+t'h') - f(x+th) - f(x+t'h') + f(x)].$$

Some five theorems are proved concerning direct second-order differentiability. Among these are: Fréchet implies Gâteaux; if f has a Gâteaux derivative which is zero, then for all x, y, $f(x+y) = f(x) + f(y) - f(0)$; if f is Fréchet differentiable at (x_1, x_2), then it is Fréchet differentiable (in the ordinary sense) separately at x_1 and at x_2.

E. R. Lorch (New York, N.Y.)

CALCULUS OF VARIATIONS

See also 5701.

5920:

Graves, Lawrence M. A multiplier rule for a class of two-dimensional variational problems. Bull. Amer. Math. Soc. **65** (1959), 374–376.

The author formulates a class of two-dimensional variational problems and obtains for them a multiplier rule. The function to be minimized depends upon a set of integrals, whose integrands depend upon surfaces in non-parametric form as well as their first and second partial derivatives. Various boundary conditions on the surfaces are imposed as well as a set of side conditions in the form of partial differential equations of the second order. The multiplier rule is then established under an assumption on the side conditions. The proofs are given in two technical reports by the author to the Office of Ordnance Research.

H. H. Goldstine (Yorktown Heights, N.Y.)

GEOMETRY

See also 5551, 5627, 5641, 5942.

5921:

★Royden, H. L. Remarks on primitive notions for elementary Euclidean and non-Euclidean plane geometry. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 86–96. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

The author considers axiom systems P, P^* for euclidean, E, E^* for elliptic, H, H^* for hyperbolic plane geometry, couched in terms of order (betweenness, resp. separation) and equidistance (congruence of segments). P is equivalent to Hilbert's system without the axioms of completeness and of Archimedes; E, H are the corresponding modifications for the non-euclidean cases. The starred systems are obtained by adding to the unstarred ones an axiom to the effect that any line through a point inside a circle meets the circle. P, E, H define the geometries coordinizable by means of a Pythagorean field (an ordered field in which the sum of two squares is a square), whilst in the case of P^*, E^*, H^* the field is Euclidean (every positive element is a square). The investigation is concerned with the definability of certain relations in terms of others within the restricted predicate calculus. The main results are as follows. Order can be defined in terms of collinearity in E^*, P^*, and H; it cannot be defined in E and P in terms of collinearity and equidistance. In E, H, and P collinearity and equidistance can be defined in terms of orthogonality (a ternary relation expressing that x, y, z are the vertices of a triangle with a right angle at x). Thus in E^*, P^* and H orthogonality can be used as the sole primitive notion. Orthogonality does not suffice for the definition of order in E and P. In H^* collinearity can be taken as the sole primitive notion. The binary relation of two points being at the polar distance (one-half the length of the straight line) can be used as the sole primitive notion in E^* but not in E. In E the binary relation of two points being closer than half the polar distance may be used as the sole primitive.

F. A. Behrend (Melbourne)

5922:

★Robinson, Raphael M. Binary relations as primitive notions in elementary geometry. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 68–85. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

As in the paper reviewed above, euclidean, elliptic and hyperbolic geometry are given in terms of order and equidistance; any dimension $p \geq 2$ is permitted, but only the standard case is considered where the coordinate field is the field of all real numbers. M. Pieri has shown that the ternary relation of a point being equidistant from two other points can be used as the sole primitive in the Euclidean case; the author shows that this holds for the non-Euclidean cases as well. The question is raised whether one or more binary relations can serve as primitive notions. Clearly, this cannot be the case in Euclidean geometry, as a similarity preserves the given primitives of order and equidistance but does not preserve any non-trivial binary relation. But the problem can be restored by adding, in the Euclidean case, a unit-distance and the binary relation of two points being a unit-distance apart. It is shown that the binary relation of two points being at distance $\pi/2$ may be used as the only primitive in the elliptic case (a result also obtained by Royden, see the preceding review), but that no finite number of binary primitives suffices in the hyperbolic and in the Euclidean case (with unit

distance). There are further results on the local definability of equidistance in terms of a single given distance, and the definability of distances in terms of a given distance with or without the use of equidistance.

F. A. Behrend (Melbourne)

5923:

Thébault, Victor. Geometry of trihedrons. Amer. Math. Monthly **66** (1959), 496–497.

If $T=ABCD$, $T'=A'B'C'D'$ are a Moebius pair of tetrahedrons, the isogonal conjugate lines, for T [T'], of the lines AA', BB', CC', DD', meet the respective faces of T [T'] in the vertices of a tetrahedron forming a Moebius couple with T [T']. In particular, the isogonal conjugates of the vertices A', B', C', D', of T' for the triangles BCD, CDA, DAB, ABC, respectively, are the vertices of a tetrahedron forming a Moebius couple with the tetrahedron T. The latter proposition remains valid if the four points A', B', C', D', are collinear. [Cf. Mathesis **68** (1959), 206–208.]

N. A. Court (Norman, Okla.)

5924:

Thébault, Victor. Sphères associées à un tétraèdre. Amer. Math. Monthly **66** (1959), 783–789.

Given a tetrahedron $T=ABCD$, let (DA) denote the sphere tangent at the vertex D to the face BCD and passing through the vertices D and A; (D) the triad of spheres (DA), (DB), (DC); $(BCD\text{-}DA)$ the sphere tangent at D to the edge DA and circumscribed to the face BCD; $(\mathscr{D})$ the triad of spheres $(DBC\text{-}DA)$, $(DCA\text{-}DB)$, $(DAB\text{-}DC)$. With each of the three vertices A, B, C is associated a triad of spheres analogous to (D); one analogous to $(\mathscr{D})$ is associated with each set of three edges (AB, AC, AD), (BA, BC, BD), (CA, CB, CD).

The author proves a number of properties of the spheres thus defined. Here are two of the simpler ones: 1. The radical axis of one of the triads of type (D) coincides with the radical axis of one of the triads of type $(\mathscr{D})$; the four lines thus obtained form a hyperbolic group; the four lines are concurrent if T is isodynamic, the common point being the second Lemoine point of T. 2. The radical center of four spheres passing through the same three vertices and tangent to the same edge or to the opposite edge, lies on the bimedian relative to the two edges considered.

The properties obtained lead the author to associate with the tetrahedron twenty-four points which may be considered as the analogs, for the tetrahedron, of the Brocard points of the triangle.

N. A. Court (Norman, Okla.)

5925:

Baier, Othmar. Über die Konstruktion von Parallelen mit Hilfe des Lineals und ortsfester rechter Winkel. Math. Z. **71** (1959), 94–98.

Construction of a parallel is impossible with a ruler alone. Referring to H. Lebesgue [*Leçons sur les constructions géométriques*, Gauthier-Villars, Paris, 1950; MR **11**, 678] in his last lecture, the problem and solution of constructing a parallel line in Euclidean 2-space with a minimum of means is enriched by investigating the construction for the case (1) that a ruler (even a short one) and p, p_1 and q, q_1 (two pairs of parallels) are available, and (2) that a ruler and a right angle are available for any arbitrary vertex. Known theorems of projective geometry provide the insight that in case (1) construction of parallels is always feasible; in case (2), in general, not.

Three non-collinear points, each the vertex of two right angles, and (even a short) ruler suffice for the construction of two pairs of parallel lines, and no fewer pairs of right angles will do. Moreover, two points, each the vertex of a triple of equal (not right) angles provide the means to construct parallels.

S. R. Struik (Cambridge, Mass.)

5926:

Tănăsescu, Aurelian. Note sur la ponctuation des épures d'intersection des polyèdres et des surfaces. Bul. Inst. Politehn. Bucureşti **20** (1958), no. 2, 55–60. (Russian, English and German summaries)

The author describes a method which enables the beginner to decide mechanically which parts of the vertical projection of the intersection of polyhedra and surfaces are "visible".

F. A. Behrend (Melbourne)

5927:

Havel, Václav. Fundamentalsätze der mehrdimensionalen Zentralaxonometrie. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **7** (1957), 94–107. (Czech. Russian and German summaries)

The author finds a necessary and sufficient condition for a configuration $O, A_1, \cdots, A_n, B_1, \cdots, B_n$ in $(n-1)$-dimensional space to be the projection of a "coordinate configuration" consisting of an origin O', the unit-points $A_1', \cdots, A_n'$ and the points at infinity $B_1', \cdots, B_n'$ of a set of n orthogonal axes through O' in n-dimensional space (first fundamental theorem); this generalizes results of E. Kruppa and N. F. Četveruhin [Dokl. Akad. Nauk SSSR **50** (1945), 75–76; MR **14**, 575]. Generalizations are also given of the second fundamental theorem [see, e.g., Stiefel, *Lehrbuch der darstellenden Geometrie*, Birkhäuser, Basel, 1947; MR **9**, 155].

F. A. Behrend (Melbourne)

5928:

Sherk, F. A. The regular maps on a surface of genus three. Canad. J. Math. **11** (1959), 452–480.

Die regelmässigen Überdeckungen $\{p, q\}$ der Ebene sind seit langem bekannt und werden von p-Ecken, deren je q an einer Ecke zusammenstossen, geliefert. Die zugehörige Gruppe wird von zwei Elementen R und S erzeugt, wobei gilt

$$(*) \qquad R^p = S^q = (RS)^2 = E.$$

Ebenso ist das analoge Problem für Flächen vom Geschlechte eins und zwei gelöst, während für Flächen vom Geschlechte drei nur Teilresultate bekannt sind. Verf. zeigt, dass es in diesem Falle zwölf verschiedene regelmässige Überdeckungen gibt, die sowohl übersichtlich in Tabellenform mit den zugehörigen Gruppen zusammengestellt sind wie auch durch Figuren erläutert werden. Die Methode zur Aufzählung beruht darauf, dass in den regelmässigen Ebenenteilungen gewisse Flächen identifiziert werden, was bewirkt, dass zu den obigen Relationen (*) weitere hinzutreten. Neben obigem Hauptergebnis findet der Verf. u.a. die zehn regelmässigen Überdeckungen des Typus $\{p, 3\}$ mit sechs oder weniger Flächen.

J. J. Burckhardt (Zürich)

5929:

Emde, Helmut. Homogene Polytope. Bayer. Akad. Wiss. Math.-Nat. Kl. Abh. **89** (1958), 67 pp.

The author's "homogeneous polytope" is a 3-complex consisting of a topologically regular partition of a 3-sphere into cells; that is, each cell is a topologically regular polyhedron $\{n, q\}$, bounded by n-gons, q at each vertex, and every edge of the complex belongs to the same number of cells, say m. Thus the homogeneous polytopes include the spherical honeycombs $\{n, q, m\}$ [Coxeter, *Regular polytopes*, Methuen, London, 1948; MR **10**, 261; pp. 135, 137], which exist whenever n, q, m are integers, greater than 1, satisfying

$$\sin\frac{\pi}{n}\sin\frac{\pi}{m} > \cos\frac{\pi}{q}.$$

As they also include the very simple complexes

$$\{n, 2, 1\}, \quad \{2, 1, 2\}, \quad \{1, 2, m\},$$

the possible values of n, q, m are precisely the positive integers for which there is a finite group defined by

$$\begin{aligned} T_1^n = T_2^q = T_3^m = (T_1T_2)^2 &= (T_1T_2T_3)^2 \\ &= (T_2T_3)^2 = 1 \end{aligned}$$

[J. A. Todd, Proc. Cambridge Philos. Soc. **27** (1931), p. 217]. This is the group of sense-preserving automorphisms of the complex.

Regarding the complex as a configuration having N_j j-dimensional elements ($j=0, 1, 2, 3$), each incident with N_{jk} k-dimensional elements, we have $N_jN_{jk}=N_kN_{kj}$. Since $N_{20}=N_{21}=n$, $N_{10}=N_{23}=2$, $N_{12}=N_{13}=m$, $nN_{32}=2N_{31}=qN_{30}$, $qN_{03}=2N_{02}=mN_{01}$, $N_{30}+N_{32}=N_{31}+2$, $N_{01}+N_{03}=N_{02}+2$, we have also

$$N_0N_{02} = nN_2 = mN_1 = N_3N_{31}, \quad N_{30}N_{02} = N_{03}N_{31},$$

$$N_0 : N_1 : N_2 : N_3 = \frac{1}{q}+\frac{1}{m}-\frac{1}{2} : \frac{1}{m} : \frac{1}{n} : \frac{1}{n} : +\frac{1}{q}-\frac{1}{2}$$

[*Regular polytopes*, p. 132]. The author obscures the simplicity of these relations by refusing to use the Schläfli symbol $\{n, q, m\}$, although he refers to Schläfli in his bibliography. Regarding the 3-sphere as a conformal 3-space, he gives beautifully shaded drawings, and pairs of photographs of solid models ready for use with a stereoscope, taking the figures in the order $\{n, 2, m\}$, $\{n, 2, 1\}$, $\{n, 2, 2\}$, $\{1, 2, m\}$, $\{2, 2, m\}$, $\{2, 2, 2\}$, $\{2, n, 2\}$, $\{2, 3, 3\}$, $\{2, 4, 3\}$, $\{2, 5, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, $\{3, 3, 2\}$, $\{3, 3, 3\}$, $\{3, 3, 4\}$, $\{3, 3, 5\}$, $\{3, 4, 2\}$, $\{3, 4, 3\}$, $\{3, 5, 2\}$, $\{4, 3, 2\}$, $\{4, 3, 3\}$, $\{5, 3, 2\}$, $\{5, 3, 3\}$. In the last case, the wire model with beads at its 600 vertices is probably the finest model of the 120-cell that has ever been made. This is supplemented by a plaster model in which the cells can be inserted one by one.

H. S. M. Coxeter (Cedar Falls, Iowa)

5930:

★**Maxwell, E. A. General homogeneous co-ordinates in space of three dimensions.** Cambridge University Press, New York, 1959. xiv+169 pp. $2.75.

Reprinting of the 1951 edition [MR **12**, 731].

5931:

Straszewicz, S. Second degree curves in elementary analytic geometry. Wiadom. Mat. (2) **1** (1955/56), 234–243. (Polish)

5932:

★**Maisano, Francesco. Sulla struttura dei piani liberi di M. Hall.** Convegno internationale: Reticoli e geometrie proiettive, Palermo, 25–29 ottobre 1957; Messina, 30 ottobre 1957, pp. 87–98. (1 insert) Edito dalla Unione Matematica Italiana con il contributo del Consiglio Nazionale delle Ricerche. Edizioni Cremonese, Rome, 1958. vii+141 pp. 1800 Lire.

Although the title of this paper mentions M. Hall [Trans. Amer. Math. Soc. **54** (1943), 229–277; MR **5**, 72], the only specific reference is to a paper by L. Lombardo-Radice [Rend. Sem. Mat. Univ. Padova **24** (1955), 312–345; MR **17**, 776]. The free (projective) plane $P=P^p$ (where $p\geqq 4$ is a fixed integer) may be regarded as the union of "point-stages" π_n or "line-stages" ρ_n ($n>0$). Here π_1 is the free partial plane consisting of two distinct lines L_1, L_2, two points incident with L_1 but not with L_2, and $p-2$ points incident with L_2 but not with L_1. If π_n has been defined, ρ_n is obtained from π_n by freely adjoining the missing joins of pairs of points of π_n, and π_{n+1} is obtained from ρ_n by freely adjoining the missing intersection points of pairs of lines of ρ_n. According to Lombardo-Radice, a point (or line) of P has degree n if it is in π_n (or ρ_n) but not in π_k (or ρ_k) for $k<n$.

The present author associates with each point of degree n a sequence of non-negative integers of length $2^{2n}-2$, arranged in $2n-1$ successive blocks of lengths $2, 2^2, \cdots, 2^{2n-1}$ respectively; and similarly for lines of degree n, with $2n$ replaced by $2n+1$. The sequence is intended to tell the "history" of the point or line; however, an element (point or line) may be associated with more than one sequence, and the same sequence may be associated with more than one element. One example will indicate the idea and the difficulties: A line L of degree n (where L is distinct from L_1, L_2 in case $n=1$) is uniquely determined as the join of a point A of degree n and a point B of degree $j\leqq n$; if $j=n$, the roles of A, B may be interchanged. Suppose that sequences for A, B have been determined. The first block (of length 2) in the sequence for L is n, j. For $1\leqq k\leqq 2n-1$, the $(k+1)$th block in the sequence for L begins with the kth block in the sequence for A and ends: (1) if $k\leqq 2j-1$, with the kth block in the sequence for B; (2) if $2j-1<k\leqq 2n-1$, with a block of length 2^k made up of zeros and ones, depending on the "history" of B. {Reviewer's remark: The difficulties occur in formulating rules for assigning sequences to elements of degree 1 and in giving precise rules for (2) above. The author gives one set of rules in the paper and another set in an unnumbered insert following p. 98.}

Two elements may be said to have the same type if they are associated with the same class of sequences. The rest of the paper is devoted to finding (inductive) formulas for the number of types of each degree and the number of elements of each type.

R. H. Bruck (Ospedaletti)

5933:

Hall, Marshall, Jr.; Swift, J. Dean; and Killgrove, Raymond. On projective planes of order nine. Math. Tables Aids Comput. **13** (1959), 233–246.

The four known projective planes of order nine (the desarguesian plane; the plane coordinizable by a Veblen-Wedderburn system which is not a field; its dual; and a self-dual non-desarguesian plane discovered by Veblen and Wedderburn) all have an elementary abelian addition in an

appropriate ternary ring. It was shown by a search made partly by hand, partly by machine, that no further planes exist with this property, but that the three non-desarguesian planes all may be coordinatized in more than one way with an abelian addition. The result strengthens the conjecture that the four planes are a complete set for order nine, but a complete search was impossible within the limits of the available equipment.

F. A. Behrend (Melbourne)

5934:

Kleinfeld, Erwin. Finite Hjelmslev planes. Illinois J. Math. **3** (1959), 403–407.

Als Hjelmslev-Ebene bezeichnet Verf. eine projektive Ebene mit Nachbarelementen im Sinne von W. Klingenberg [Math. Z. **60** (1954), 384–406; Abh. Math. Sem. Univ. Hamburg **20** (1955), 97–111; MR **16**, 507; **17**, 522] ohne Voraussetzung des desarguesschen Satzes, d.h. ein System von Punkten und Geraden mit einer Inzidenz-Beziehung, so daß je zwei Punkte mit mindestens einer Geraden und je zwei Geraden mit mindestens einem Punkt inzidieren und die Klassen benachbarter Punkte und Geraden eine projektive Ebene im gewöhnlichen Sinne bilden. Dabei heißen zwei verschiedene Punkte bzw. Geraden benachbart, wenn sie mit mindestens zwei verschiedenen Geraden bzw. Punkten inzidieren. Verf. beweist für endliche Hjelmslev-Ebenen durch Abzählungen die folgenden Tatsachen: Die Punkte jeder Geraden zerfallen in $n+1$ Klassen von je t Nachbarpunkten. Die Anzahl aller Punkte sowie die Anzahl aller Geraden ist $(n^2+n+1)t^2$. Wenn $t>1$ ist, so ist $n\leqq t$. Die durchschnittliche Zahl von Schnittpunkten einer Geraden mit einer von ihr verschiedenen Nachbargeraden ist $\lambda=(n+1)t(t+1)^{-1}$. Im allgemeinen ist λ keine ganze Zahl; ist λ ganz, so ist sogar $\lambda=t=n$ und das Produkt der Inzidenzmatrix mit ihrer Transponierten ist eine Matrix mit Elementen t^2+t in der Hauptdiagonale, t in quadratischen Blöcken von t^2 Reihen um die Hauptdiagonale herum und 1 sonst. Für $n=2$ und $t=3$ gibt es keine Hjelmslev-Ebene. Schließlich gibt Verf. in Beantwortung einer von Klingenberg [loc. cit.] aufgeworfenen Frage durch Konstruktion der entsprechenden nicht kommutativen Koordinatenbereiche für jede Primzahl p und jedes $k\geqq 2$ Beispiele endlicher desarguesscher aber nicht pappusscher Hjelmslev-Ebenen mit $t=n=p^k$. Dabei ist ein sinnstörender Druckfehler unterlaufen: Die Koordinaten-Ringe H desarguesscher Hjelmslev-Ebenen sind nach Klingenberg dadurch gekennzeichnet, daß in ihnen die Nicht-Nullteiler invertierbar sind und die Nullteiler ein Ideal N bilden derart, daß es zu n, n' aus N stets x, y aus H (nicht notwendig aus N!) gibt mit $n=xn'$ oder $n'=xn$ und $n=n'y$ oder $n'=ny$.

H. Salzmann (Frankfurt/Main)

CONVEX SETS AND GEOMETRIC INEQUALITIES

See also 5795, 5882, 5883, 6062.

5935:

Kárteszi, Ferenc. Sur les figures convexes enveloppées par des carrées. Köz. Mat. Lapok **18** (1959), 1–6, 33–37. (Hungarian)

The author considers the following problem: Characterize all closed convex curves (polygons) with the property that all circumscribed rectangles are squares. He proves that the sum (in the sense of Minkowski) of such curves is again such a curve (the square and all curves of constant width are examples of such curves). He further shows that a polygon with an angle $<\pi/2$ can not have the above property, but does not give a complete characterization of the polygons all of whose circumscribed rectangles are squares.

P. Erdös (Adelaide)

5936:

Goldberg, Michael. Intermittent rotors. J. Math. Phys. **38** (1959/60), 135–140.

It was proved independently by D. B. Sawyer [Quart. J. Math. Oxford Ser. (2) **4** (1953), 282–292; MR **15**, 607] and J. J. Schäffer [Math. Ann. **129** (1955), 265–273; MR **16**, 1145] that the closed set of minimal area which contains in any position at least one vertex of a given square lattice {4, 4} consists of a square with segments of parabolas added on opposite sides, the parabolic arcs making angles of 45° with these sides. The author regards this set as a rotor which can be turned so that its boundary continually passes through at least three of the four vertices of a fixed square. He points out that, if we abandon the requirement of minimal area, the parabolic arcs can be replaced by any centrally symmetrical pair of arcs lying closer to the sides of the square. Then the remaining two "sides" of the rotor cease to be straight; for instance, in the case of circular arcs they belong to certain sextic curves. {In view of the amount of arbitrariness, the reader naturally wonders whether, by a suitable choice, the four arcs could be united to form a closed curve having a single equation involving elementary functions.} The author considers various generalizations of the problems, such as replacing the fixed square by other polygons.

H. S. M. Coxeter (Cedar Falls, Iowa)

5937:

Fejes Tóth, L. On the sum of distances determined by a pointset. Acta Math. Acad. Sci. Hungar. **7** (1956), 397–401. (Russian summary)

The author proves: (1) the sum S_n of the $\binom{n}{2}$ distances determined by $n\geqq 2$ coplanar points satisfies the inequality $S_n\leqq rn\operatorname{ctg}(\pi/2n)$, where r denotes the circumradius of the points, and equality holds only if the points are the vertices of a regular n-gon; (2) if S_n denotes the sum of the mutual distances of n points of the $(n-1)$-dimensional Euclidean space, then

$$S_n \leqq n\binom{n}{2}^{\frac{1}{2}} r,$$

where r is the circumradius of the points. Equality holds only for the vertices of a regular simplex.

L. M. Blumenthal (Columbia, Mo.)

5938:

Sz.-Nagy, Béla. Über Parallelmengen nichtkonvexer ebener Bereiche. Acta Sci. Math. Szeged **20** (1959), 36–47.

Let G be a closed, non-empty, proper subset of the plane, G^* its complement. For real $t\geqq 0$, let G_t denote the union of all closed circular discs with diameter t and centers in G. G is of type (n,ν) (n a non-negative integer, $\nu=0$ or 1) if it consists of n bounded and ν unbounded

connected components. $\rho^*(G)$ is the supremum of the radii of discs contained in G^*. Similarly, for open sets G and $t \geq 0$ let $G_{-t} = ((G^*)_t)^*$, and let $\rho(G)$ be the supremum of the radii of discs contained in G. The main results are the following. Theorem 1: Let G be of type (n, ν); then $A(G_t - G) - \pi(n-\nu)t^2$ is a continuous, concave function of t in the interval $0 \leq t < \rho^*(G)$ ($A(S)$ denotes the Lebesgue measure of S). Theorem 2: Let G be an open set whose complement is of type (n, ν); then $A(G - G_{-t}) - \pi(n-\nu)t^2$ is a continuous, concave function of t in the interval $0 \leq t < \rho(G)$. As corollaries the author obtains inequalities generalizing Steiner's formula $L(G_t) = L(G) + 2\pi t$ (G convex, $L(G)$ being the length of Bd G) to sets of types (n, ν).
B. Grünbaum (Princeton, N.J.)

5939:

Pestov, G.; and Ionin, V. On the largest possible circle imbedded in a given closed curve. Dokl. Akad. Nauk SSSR **127** (1959), 1170–1172. (Russian)

The main result is: Let C denote a simple, closed Jordan curve in the plane, twice continuously differentiable, such that the radius of curvature at each point of C is not less than r_0. Then there exists a circular disc of radius r_0 contained in the union of C and the bounded domain determined by C. This theorem and some generalizations are established using simple lemmata.
B. Grünbaum (Princeton, N.J.)

5940:

Lagunov, V. N. On the largest possible sphere imbedded in a given closed surface. Dokl. Akad. Nauk SSSR **127** (1959), 1167–1169. (Russian)

Let F_r denote a closed, twice continuously differentiable surface in E^3, whose principal radii of curvature in each point are not less than r. Let K_r denote the closure of the bounded component of the complement of F_r. Theorem: For each F_r there exists a ball of radius αr, $\alpha = -1 + \frac{2}{3}\sqrt{3}$, contained in K_r. For each $\varepsilon > 0$ there exists an F_r such that K_r contains no ball of radius $\alpha r + \varepsilon$. The note contains a proof of the first part of the theorem. The proof of the second part, as well as those of some generalizations, are said to be too complicated to permit inclusion; they will be published subsequently. *B. Grünbaum* (Princeton, N.J.)

5941:

Heppes, Aladár. An extremal property of the spherical net of the cuboctahedron. Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1/2, 97–99. (Hungarian. Russian and English summaries)

n great circles always divide the sphere into $n(n-1)+2$ domains if some domains are allowed to have zero area. When investigating the subdivision of a sphere by great circles, two questions are natural: When is the area of the domain(s) of smallest area maximal? When is the area of the domain(s) of greatest area minimal? The author proves that if the number of great circles is 4 then the area of the domain of smallest area is maximal if and only if the four great circles form the spherical net of a cuboctahedron. He also proves that the area of the domain of greatest area is not minimal for this configuration of great circles, by comparing it with the configuration formed by three great circles through the north and south poles dividing the sphere into six equal areas, together with the equator. The paper is fully illustrated by diagrams.
G. A. Dirac (Hamburg)

5942:

★**Blumenthal, Leonard M. New metric postulates for elliptic n-space.** The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 127–145. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

The author has shown [Trans. Amer. Math. Soc. **59** (1946), 381–400; Pacific J. Math. **5** (1955), 161–167; MR **8**, 82; **16**, 1139] that the following postulates serve to characterize metrically elliptic n-space. (E_r denotes a distance space, pq the distance between points p and q, and pqr means $pq+qr=pr$.) (1) E_r is semimetric. (2) E_r is metrically convex. (3) The diameter of E_r is at most $\pi r/2$; (4) E_r is metrically complete. (5) If $p, q \in E_r$ and $pq \neq \pi r/2$, then E_r contains points p^* and q^* such that pqp^*, qpq^* and $pp^* = qq^* = \pi r/2$. (6) If p_0, p_1, p_2, p_3, p_4 are any five pairwise distinct points of E_r with (a) two linear triples, (b) the determinant $\Delta^* = |\cos(p_ip_j/r)|$ of three of the points (one of which is common to the two linear triples) negative, then an ε matrix (ε_{ij}) $(i, j = 0, 1, 2, 3, 4)$ exists such that all principal minors of the determinant $|\varepsilon_{ij}\cos(p_ip_j/r)|$ are non-negative. This axiom means, in essence, that any quintuple of a prescribed subclass of those containing two linear triples can be congruently imbedded in elliptic space. (7) A postulate fixing the dimension which need not concern us here.

This paper, presented at the Berkeley symposium on the axiomatic method in Dec. 1957, gives several weakenings of these axioms by considering various alternatives to (6). Thus (6) may be replaced by (6_w): Each four points of E_r containing a linear triple is congruently imbeddable in elliptic space. This can be expressed in terms of the determinant Δ^* mentioned above. In this event E_r is said to have the elliptic weak four point property.

Or if (1) is replaced by (1_m) requiring E_r to be metric then (6) can be replaced (6_f): Each four points of E_r containing an isosceles linear triple (i.e., three points p_1, p_2, p_3 with $p_1p_2 = p_2p_3$ and $p_1p_2 + p_2p_3 = p_1p_3$) is congruently imbeddable in elliptic space.

Various other four point conditions are considered bearing the names (a) elliptic feeble, (b) isosceles feeble, (c) ε-feeble, (d) external isosceles feeble.

Attention is called to the fundamental unsolved problem of the congruence order of elliptic n-space $n > 2$.
L. M. Kelly (East Lansing, Mich.)

GENERAL TOPOLOGY, POINT SET THEORY

See also 5569, 5705.

5943:

Ivanova, V. M.; and Ivanov, A. A. Contiguity spaces and bicompact extensions of topological spaces. Dokl. Akad. Nauk SSSR **127** (1959), 20–22. (Russian)

The authors are concerned with faithful T_1 compactifications of a space E, that is, those for which the closures of subsets of E form a basis of closed sets. They describe them by means of predicates σ applied directly to finite families α of closed sets of E. The axioms are: (C1) If α is a refinement of β and $\sigma(\alpha)$, then $\sigma(\beta)$; (C2) if $\alpha=\{A_i\}$, $\beta=\{B_j\}$, and $\sigma(\{A_i \cup B_j\})$, then $\sigma(\alpha)$ or $\sigma(\beta)$; (C3) for a point x and a closed set F, $\sigma(x, F)$ if and only if $x \in F$. A predicate satisfying these axioms is called a contiguity.

J. Isbell (Lafayette, Ind.)

5944:

Gál, I. S. Proximity relations and pre-compact structures. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math **21** (1959), 304–326.

A proximity relation is a symmetric and anti-reflexive binary relation $\wedge$ on the collection of subsets of a set X which has the properties: $A \wedge (B \cup C)$ if and only if $A \wedge B$ and $A \wedge C$, and if $A_1 \wedge A_2$ then $A_i \wedge cB_i$ $(i=1, 2)$ for suitable B_i satisfying $B_1 \wedge B_2$. (cA denotes the complement of A.) A leisurely exposition of some of the theory of these structures is given here, together with a short introduction to the theory of uniform structures. This contains a generalization to the non-separated (non-Hausdorff) case of a theorem of Yu. M. Smirnov [Dokl. Akad. Nauk SSSR **84** (1952), 895–898; MR **14**, 1107], the theorem being that there is natural one-to-one correspondence between proximity relations on X and precompact uniform structures on X. The author also gives an account of the proximity relations related to some special precompact uniform structures of H. Freudenthal [Fund. Math. **39** (1952), 189–210; MR **14**, 893].

H. H. Corson (Seattle, Wash.)

5945:

Isiwata, Takesi. On locally Q-complete spaces. I. Proc. Japan Acad. **35** (1959), 232–236.

Suppose X is not a Q-space, i.e., not complete in the uniform structure generated by the set $C(X)$ of all real-valued continuous functions on X. Then the following conditions are equivalent: (1) X is locally complete in the uniform structure generated by $C(X)$; (2) X is open in its completion in this structure; (3) X is dense in a Q-space obtained by adjoining one point. A similar theorem relating local compactness and local completeness is also given. {Several of the proofs contain errors. The results are correct except for the non-trivial half of Theorem 3; however, this is applied only to dense subspaces, for which it is true.}

C. W. Kohls (Urbana, Ill.)

5946:

Isbell, J. R. Embeddings of inverse limits. Ann. of Math. (2) **70** (1959), 73–84.

The paper deals with uniform spaces and characterizes those spaces X which are uniformly equivalent to subspaces of Euclidean spaces E^n (E^n provided with the usual metric uniformity) by means of these four conditions: (1) X has a countable basis of uniform coverings (u.c.), i.e., its uniformity is metric; (2) X has a basis of u.c. whose nerves are p-dimensional; (3) X has a star-bounded basis of u.c. $\{u^\alpha\}$, i.e., for each pair α, β there is a natural number $a_{\alpha\beta}$ such that no element of u^α meets more than $a_{\alpha\beta}$ elements of u^β; (4) X has a basis of u.c. whose nerves have Euclidean 1-skeletons [in the sense of the author's paper, Pacific J. Math. **8** (1958), 67–86; MR **20** #4261]. As the author points out, this result should be completed by a characterization of Euclidean 1-complexes in terms of properties depending on the incidence relations alone.

The technique of the proofs depends on expanding metric uniform spaces X into inverse systems of the nerves of some u.c. of X (lemma 1) and on discussing conditions when the inverse limit of spaces embeddable in a space E is itself embeddable in E. Incidentally, two purely topological results are obtained: Th. 1: An inverse limit of a sequence of compact subspaces of E^n is a subspace of E^{2n}; Th. 2: An inverse limit of a sequence of subspaces of E^1 is a subspace of E^2.

S. Mardešić (Zagreb)

5947:

Corson, H. H. Normality in subsets of product spaces. Amer. J. Math. **81** (1959), 785–796.

A Σ-product of spaces X_a $(a \in A)$ with base-point $p=(p_a)$ is the dense subspace Σ of $\prod X_a$ consisting of those points $x=(x_a)$ for which $x_a=p_a$ for all but countably many a's. If "countably" is replaced by "finitely", Σ is called a σ-product. The principal results include the following. Assume throughout that each X_a is metrisable. If each X_a is complete, the Σ-product Σ is collectionwise normal and countably paracompact. If each X_a is separable, the real-compactification $\nu\Sigma$ of Σ is $\prod X_a$; by taking A uncountable and X_a complete but not compact, one obtains a normal space Σ whose real-compactification is not normal, answering a question of Gilman and Jerison. A particular subspace F_0 of the Σ-product of countable discrete spaces is not paracompact although the family of all neighborhoods of the diagonal in $F_0 \times F_0$ forms a complete uniformity for F_0; this disproves a conjecture of Kelley [*General topology*, Van Nostrand, New York, 1955; MR **16**, 1136; pp. 208–209], for which a previously announced counterexample was erroneous. No Σ-product of uncountably many non-trivial spaces can be completely normal. When each X_a is the real line, Σ is homeomorphic to the space of all continuous real functions on some Lindelöf space. If each X_a is separable, their σ-product is Lindelöf; if further each X_a is σ-compact (the author assumes more than this, but his argument appears to generalize) then so is their σ-product.

A. H. Stone (Manchester)

5948:

Papić, Pavle. Sur les espaces de Baire généralisés. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II **14** (1959), 7–12. (Serbo-Croatian summary)

The author is interested in subspaces of countable products of discrete spaces of given cardinal. He shows that every metrizable space has one of these as a dense subspace, and is also a continuous image of one. The cardinal he uses is the maximum cardinal of closed discrete subsets, which is not always best possible.

J. Isbell (Lafayette, Ind.)

5949:

Plunkett, Robert L. Concerning two types of convexity for metrics. Arch. Math. **10** (1959), 42–45.

A subset of a metric space which is isometric with a closed and bounded interval of real numbers is called a segment. A metric for which there exists at most one segment joining each pair of points is called a convex

metric with unique segments. A metric d for a space X is called hyperconvex if and only if, for each class of closed spheres $\bar{S}(x_i, r_i)$, $i \in I$, where I is an indexing set, satisfying the condition that $d(x_i, y_j) \leq r_i + r_j$ for all i and j of I, it is true that $\bigcap_{i \in I} \bar{S}(x_i, r_i) \neq \emptyset$. The author proves that if X is a compact space with metric d then any two of the following statements imply the remaining one: (α) d is a convex metric with unique segments; (β) d is a hyperconvex metric; (γ) X is a dendrite.

W. W. S. Claytor (Washington, D.C.)

5950:

Ellis, Robert. Equicontinuity and almost periodic functions. Proc. Amer. Math. Soc. **10** (1959), 637–643.

Let X be a separated uniform space, let $C(X, X)$ denote the set of all continuous functions on X to X, and let $C(X)$ denote the set of all real-valued continuous functions on X. Generalizing the von Neumann definition of almost periodic functions on a group, the author defines $f \in C(X)$ to be almost periodic with respect to $A \subset C(X, X)$ provided that the set of all translates of f by elements of A is a relatively compact subset of $C(X)$ in the topology of uniform convergence. He proves that, in various cases, an abundance of almost periodic functions with respect to A implies A is equicontinuous. In particular, the author extends Baum's characterization of equicontinuous transformation groups and Maak's theorem that left and right almost periodic are equivalent. The weakly almost periodic functions of Eberlein are also generalized and discussed in similar fashion.

W. H. Gottschalk (Philadelphia, Pa.)

5951:

Pearson, B. J. A connected point set in the plane which spirals down on each of its points. Duke Math. J. **25** (1958), 603–613.

The author shows that if, in the plane, ABC is a triangle, there exists a collection G of arcs such that (1) each of them has A as an end-point and no two have any other point in common, (2) every straight interval with one end-point A and the other on the straight interval BC is a subset of some arc of G and each arc of G contains one such interval, (3) each arc of G spirals down on one of its end-points and on no other point, and (4) the set of all whirl points of arcs of G is a compact, connected inner limiting set which is connected "im kleinen" at all but a countable number of its points. In an earlier paper [Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 207–213; MR **14**, 783] R. L. Moore established the existence of a collection of spirals satisfying conditions (1) and (2) for which the set of all whirl points is perfect, but totally disconnected.

J. T. Mohat (Denton, Tex.)

5952:

Chamberlin, R. E. Tree-like continua and quasi-complexes. Duke Math. J. **26** (1959), 511–517.

A tree-like continuum is a metric continuum which admits arbitrarily fine open covers whose nerves are acyclic 1-dimensional complexes. Existence of tree-like continua which are not quasi-complexes (q.c.) is demonstrated by a T-shaped set in the plane together with a ray spiraling down to the T-set. In order to prove that this continuum fails to be a q.c., the author specializes the definition of a q.c. [S. Lefschetz, *Algebraic topology*, Amer. Math. Soc., New York, 1942; MR , 84] to the case of tree-like continua and obtains a usable criterion. Further, each tree-like continuum is shown to be the intersection of a sequence of tree-like continua which are quasicomplexes.

S. Mardešić (Zagreb)

5953:

Bing, R. H. The cartesian product of a certain nonmanifold and a line is E^4. Ann. of Math. (2) **70** (1959), 399–412.

Proofs of results announced in Bull. Amer. Math. Soc. **64** (1958), 82–84 [MR **20** #3514]. The author has described elsewhere [same Ann. **65** (1957), 484–500; MR **19**, 1187] an upper semi-continuous decomposition G of E^3 into points and tame arcs, whose decomposition space B has many strange properties. In particular, B is not a 3-manifold and so $B \neq E^3$. It is now proved that the Cartesian product of B and a line E^1 is E^4. The main idea is the perception of unexpected 4-cells enclosing the elements of G, when they are considered to lie in E^4. These 4-cells are then used to construct a pseudo-isotopy of E^4 on itself, which shrinks the elements of G to points in a controlled manner. The proof has various corollaries; e.g., if $\bar{B}$ is the one-point compactification of B, then $\bar{B} \times E^1 = S^3 \times E^1$, the suspension of $\bar{B}$ is S^4, etc. Thus, there is a homeomorphism of S^4 on itself, of period 2, whose fixed point set is $\bar{B}$, and so not a manifold. In contrast, the author sketches a proof that there is no space A different from E^2, such that $A \times E^1 = E^3$.

H. B. Griffiths (Bristol)

5954:

Bing, R. H. Conditions under which a surface in E^3 is tame. Fund. Math. **47** (1959), 105–139.

The paper is concerned with a condition that a finite or infinite surface in Euclidean 3-space E^3 be tame. Given homeomorphic subsets A, B of E^3, let $H(A, B) \leq \varepsilon$ mean that there is a homeomorphism of A on B moving no point by more than ε. Then let S be a topological 2-sphere in E^3, and denote by Int S, Ext S the bounded and unbounded components of $E^3 - S$. Theorem: If for each $\varepsilon \geq 0$ there is a topological 2-sphere S' in Int S, such that $H(S', S) \leq \varepsilon$, then $S \cup \operatorname{Int} S$ is a topological 3-cell. Applying the proof to Ext S also, one obtains the result that S is tame if for each $\varepsilon \geq 0$ there exist 2-spheres $S' \subset \operatorname{Int} S$, $S'' \subset \operatorname{Ext} S$ with $H(S', S) \leq \varepsilon$, $H(S'', S) \leq \varepsilon$. The author formulates two extensions, one by replacing S by any surface M, the other concerning local instead of global tameness; but he promises detailed proofs of these extensions in a later paper, after giving sketch proofs. In the proof of the theorem quoted above, S' can be taken polyhedral, by the author's approximation theorem in [Ann. of Math. (2) **65** (1957), 456–483; MR **19**, 300]; thus he obtains a sequence S_n of polyhedral 2-spheres with $S_n \subset \operatorname{Int} S_{n+1} \subset \operatorname{Int} S$, and $H(S_n, S) \leq 1/n$. The main problem is then to make a homeomorphism θ of $\operatorname{Int} S \cup S$ onto a 3-cell and boundary by constructing one, θ_n, on the "thick shell" between S_{n+1} and S_n; and θ_n must not stretch "radii" too far, so that $\theta | S$ is induced properly. Here the author makes a detailed analysis with many subsidiary lemmas too complicated here to state. Using "chopping" techniques, his main tool is the concept of a "fence", i.e., a product, of a linear graph in a horizontal plane, with a vertical axis. A section is included about graphs on 2-spheres.

H. B. Griffiths (Bristol)

5955:

Haupt, Otto. Verallgemeinerung eines ordnungsgeometrischen Reduktionssatzes. J. Reine Angew. Math. **200** (1958), 170–181.

Let L_k denote linear k-spaces in real projective n-space P_n. The L_{n-1}'s form the dual space R_n of P_n. If $M \subset P_n$, $f \subset R_n$, the (strong) point order of M with respect to f is the supremum of the cardinal numbers of the sets $M \cap L_{n-1}$ if L_{n-1} ranges through f. For each $L_{n-1} \in f$ form the cardinal number of the set of the components of $M \cap L_{n-1}$. The supremum of these numbers is the (strong) component order of M with respect to f. The set M is said to have a weak point [component] order a if there exists a set n nowhere dense in R_n such that M has the point [component] order a with respect to $R_n - n$. Let C be a continuum which contains $n+1$ linearly independent points and which has a weak component order $a < \infty$. C then has a component order $\leq 3a+1$ with respect to R_n and the following statements are equivalent: (i) a is a weak point order of C; (ii) C is an arc sum, i.e., the union of a countable number of Jordan arcs; (iii) C is a hereditary arc sum, i.e., every continuum contained in C is an arc sum; (iv) those L_{n-1}'s for which $C \cap L_{n-1}$ is finite are everywhere dense in R_n.

Let $L_{n-2} \subset L_{n-1}$ and let K be a component of $C \cap L_{n-1}$ which does not meet L_{n-2}. Construct a neighborhood of L_{n-1} in the linear pencil through L_{n-2} in R_n and form the point set U incident with that neighborhood. Then some component F of $C \cap (U - L_{n-1})$ will have accumulation points in K. If F is unique, K is called a non-interior supporting component. From now we also assume that C is an arc sum and reducible with respect to its order, i.e., to every pair L_k, L_{n-1} with $L_k \subset L_{n-1}$ there is an $L_{k-1} \subset L_k$ which meets all or all but one of the non-interior supporting components of $C \cap L_{n-1}$ contained in L_k $(0 \leq k \leq n-1)$. The following Reduction Theorem generalizes a result of Marchaud [Acta Math. **55** (1930), 67–115; in particular p. 79]: If $C \cap L_{n-1}$ has not less than m components, every neighborhood of L_{n-1} in R_n contains an open set o such that $L'_{n-1} \in o$ implies: $C \cap L'_{n-1}$ is finite and consists of not less than m points; all of them are intersections, i.e., there are points of C near them on both sides of L'_{n-1}. The author shows by an example that the reducibility assumption may not be dropped. Of his applications we mention only Theorem 3: a is the one and only weak point or component order of C; it is the component order of C with respect to R_n. *P. Scherk* (Toronto, Ont.)

5956:

Sanderson, D. E. Isotopy in 3-manifolds. II. Fitting homeomorphisms by isotopy. Duke Math. J. **26** (1959), 387–396.

[For part I, see Proc. Amer. Math. Soc. **8** (1957), 912–922; MR **19**, 760.] This paper bears on the useful technique of constructing homeomorphisms on a space by modifying homeomorphisms on parts of the space so that they fit together, in the following sense. Given $h: S \to R$ and $h': S' \to R'$, where (1) $h|S \cap S' = h'|S \cap S'$, (2) $S \cap S'$ separates $S \cup S'$, and (3) $R \cap R'$ separates $R \cup R'$; then the common extension of h and h' on $S \cup S'$ is a homeomorphism. The basic theorem relates to a compact polyhedral 2-manifold L on a triangulated 3-manifold M and to a piecewise linear δ-approximation f to the identity on L. It asserts that, given $\varepsilon > 0$ and a neighborhood U of L, there exists a δ so small that a simplicial ε-isotopy (deformation paths are of diameter $< \varepsilon$) exists deforming $f(L)$ pointwise onto L and leaving points on $M - U$ fixed. Various extensions are given involving manifolds with boundary. *S. S. Cairns* (Princeton, N.J.)

5957:

Kister, James. Small isotopies in Euclidean spaces and 3-manifolds. Bull. Amer. Math. Soc. **65** (1959), 371–373.

"Let M be a manifold with boundary having a metric d. Denote by $\mathscr{H}(M)$ the set of all homeomorphisms of M onto itself. Define a function ρ of $\mathscr{H} \times \mathscr{H}$ into the extended real number system as follows: $\rho(f, g) = \sup_{x \in M} d(f(x), g(x))$. f and g are ε-isotopic if there is an isotopy H_t, $t \in I$, so that $H_0 = f$, $H_1 = g$ and if $t_1, t_2 \in I$ then $\rho(H_{t_1}, H_{t_2}) \leq \varepsilon$."

This communication contains a remarkably efficient proof of theorem 1, "If f and g are in $\mathscr{H}(E^n)$ and $\rho(f, g) = \varepsilon < \infty$, then f and g are ε-isotopic", of which J. W. Alexander's theorem on the deformation of an n-cell [Proc. Nat. Acad. Sci. U.S.A. **9** (1923), 406–407] is a corollary. It also announces various results concerned with the existence of certain ε-isotopies when M is a 3-manifold with boundary. *S. S. Cairns* (Princeton, N.J.)

ALGEBRAIC TOPOLOGY

See also 5641, 5894, 5956, 5957.

5958:

Chow, Sho-kwan. Homotopy groups and cup product of cohomology groups. Sci. Sinica **8** (1959), 557–567.

English version of Acta Math. Sinica **8** (1958), 200–209 [MR **20** #5479].

5959:

Thomas, Emery. On tensor products of n-plane bundles. Arch. Math. **10** (1959), 174–179.

Let η and ζ be respectively r-plane and s-plane bundles over X. Then one may define an rs-plane bundle $\eta \otimes \zeta$ over X, and its Stiefel-Whitney classes are given by a certain polynomial $\Phi_{r,s}$ in the Stiefel-Whitney classes of η and ζ [Borel and Hirzebruch, Amer. J. Math. **80** (1958), 458–538; MR **21** #1586]. The author gives an elementary proof of this fact. To begin with, the universal example shows that the required classes are given by some such polynomials, say $\Psi_{r,s}$. Next, one can handle the case in which $r = s = 1$ and η, ζ are the standard bundles over $K(Z_2, 1)$. One can then handle suitable bundles over the product $(K(Z_2, 1))^{r+s}$, and so determine the polynomials $\Psi_{r,s}$. *J. F. Adams* (Cambridge, England)

5960:

de Lyra, C. B. On circle bundles over complex projective space. An. Acad. Brasil. Ci. **31** (1959), 17–24. (Portuguese summary)

The author gives a simple enumeration of principal SO(2)-bundles and 1-sphere bundles over complex projective space $P_n(C)$. He also proves that the base space of any locally trivial fibering of S^3 by S^1 is S^2, and that any two principal fiberings of S^3 with group SO(2) are related

by a fibre-preserving autohomeomorphism of S^3. {On p. 18 the second exact sequence should run from $H^2(L)$, which is not zero; but the conclusion $H^3(K, L)=0$ is, of course, unaffected.} *P. J. Hilton* (Birmingham)

5961:

Weier, Joseph. Ein inverser Homomorphismus der Fundamentalgruppe. Proc. Japan Acad. **35** (1959), 213–214.

Let M and N be compact, connected, orientable manifolds of dimensions $2n-1$ and n respectively ($n>2$). The author gives a construction which associates with every continuous map $f: M \to N$ a homomorphism φ of the fundamental group $\pi_1(N)$ into the integral homology group $H_n(M)$. {Reviewer's note: Apparently this homomorphism is just the composition of the natural homomorphism $\pi_1(N) \to H_1(N)$ with the Hopf "Umkehrhomomorphismus" $H_1(N) \to H_n(M)$.} *W. S. Massey* (Providence, R.I.)

5962:

Loibel, Gilberto Francisco. Multiplications on products of spheres. An. Acad. Brasil. Ci. **31** (1959), 161–162.

The author studies continuous multiplications with identity defined on a Cartesian product of spheres. He determines when such multiplications exist; when they exist, he classifies them in terms of homotopy groups of spheres. He then states, for example, that a homotopy-commutative product can exist only on a product of circles. Most of the results follow fairly easily from the known results for a single sphere.

J. F. Adams (Cambridge, England)

5963:

Swan, Richard G. A new method in fixed point theory. Bull. Amer. Math. Soc. **65** (1959), 128–130.

The paper gives results about the set of fixed points for a finite group acting on a space. The details of the proofs are not given.

Let π be a finite group acting on a space X, and A a closed π-stable subspace of X. There is a convergent spectral sequence with

$$E_2^{i,j}(X, A) = \hat{H}^i(\pi, H^j(X, A)),$$

where $\hat{H}^i$ denotes the Tate cohomology of π and $H^j(X, A)$ is the jth Čech cohomology group of (X, A) (with arbitrary coefficients) and with π action induced by the action on X. The stable term E_∞ is the graded module associated with a filtered module $J^*(X, A)$, whose description is not given in the general case. However, $E_\infty = E_2$ when π acts trivially on X.

Let X^π, A^π be the sets of fixed points in X and A respectively. Under some conditions, the inclusion map induces an isomorphism of modules (but not always of filtered modules) $J^*(X, A) \to J^*(X^\pi, A^\pi)$. If π is cyclic of prime order p, this implies

$$\sum \dim E_2^{i,j}(X^\pi, A^\pi) \leq \sum \dim E_2^{i,j}(X, A),$$

the sums being over (i, j) such that $i+j=n$, $j \geq k$, for all k and n and any coefficient group; dim means dimension as Z_p-module.

This generalizes results of E. E. Floyd [Trans. Amer. Math. Soc. **72** (1952), 138–147; MR **13**, 673] and of A. Heller [Ann. of Math. (2) **60** (1954), 283–303; MR **16**, 276] about $\sum \dim H^q(X^\pi, A^\pi; Z_p)$. A result about the cohomology rings $H^*(X^\pi; Z_p)$ and $H^*(X; Z_p)$ is given in a special case (where $A=\emptyset$; π is cyclic of prime order p acting trivially on $H^*(X; Z)$ and $H^*(X; Z_p)$; the ring $H^*(X; Z_p)$ is isomorphic to $H^*(S^{2m} \times S^{2n}; Z_p)$, where S^{2m} and S^{2n} denote spheres of even dimension).

The method uses a modification of the Cartan-Leray spectral sequence of a covering [H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, 1956; MR **17**, 1040; Chap. 16, § 8]. *G. Hirsch* (Brussels)

5964:

Andrews, J. J.; and Curtis, M. L. Knotted 2-spheres in the 4-sphere. Ann. of Math. (2) **70** (1959), 565–571.

Referring to E. Artin [Abh. Math. Sem. Univ. Hamburg **4** (1925), 174–177] and E. R. van Kampen [ibid. **6** (1927), 216], the authors give the construction of a pair of knotted 2-spheres S and S' in 4-space such that S' is homotopically linked with S, whereas S is not homotopically linked with S'. The construction runs roughly as follows: Let P be a plane in $E^3 \subset E^4$ and let E_+^3 be the half-space of E^3 above P. Let β be an arc, an overhand knot, in E_+^3 with end points in the plane P such that β gives a trefoil knot φ if its end points are joined in P by a line segment. Further let α be a planary arc in E_+^3 with end points in P such that if the end points of α are joined by a segment in P the resulting polygon gives one of the generators of the knot group of φ. If S and S' denote the 2-spheres obtained respectively by rotating β and α at the same time about the plane P, then S and S' are the required 2-spheres. To show this, it is proved that $\pi_2(S^4-S')=0$ for S' a suspension of a simple closed curve, but that $\pi_2(S^4-S) \neq 0$ for S a 2-sphere obtained by rotating an arc of the above type. Some generalizations to higher dimensions are mentioned. Question: Is $\pi_3(S^4-S)$ trivial? *H. Terasaka* (Osaka)

5965:

Kyle, R. H. Branched covering spaces and the quadratic forms of links. II. Ann. of Math. (2) **69** (1959), 686–699.

[For part I, see same Ann. **59** (1954), 539–548; MR **15**, 979]. This paper develops a cohomological dual of the theory of self-linking in the homology groups of an oriented manifold. In the dual theory, homology groups are replaced by cohomology groups and self-linking in the torsion subgroups of the former by a dual product operation $\smile_s$ in the torsion sub-groups of the latter. A method is given for calculating $\smile_s$ in the cyclic branched coverings of a tame knot or link. The method is applied to deduce some earlier results of Seifert on cyclic coverings of knots. *S. S. Cairns* (Urbana, Ill.)

DIFFERENTIAL GEOMETRY, MANIFOLDS

See also 5634, 5749, 5797, 5939, 5940, 5955.

5966:

Nomizu, Katsumi. On Frenet equations for curves of class C^∞. Tôhoku Math. J. (2) **11** (1959), 106–112.

Let $x=x(s)$ be an arc in E^3 of class C^∞ with s as arc-length. The author shows that if, for every $s=s_0$, there is an $m=m(s_0)>1$ such that $d^mx/ds^m \neq 0$ at $s=s_0$, then it is

possible to define a curvature (unique up to sign), a (unique) torsion and orthonormal (tangent, principal normal, binormal) vectors satisfying the Frenet equations. The assumption concerning the existence of $m(s)$ replaces the usual assumption that the curvature is not zero ($d^2x/ds^2 \neq 0$). *P. Hartman* (Baltimore, Md.)

5967:

Saban, Giacomo. Nuove caratterizzazioni della sfera. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. **25** (1958), 457–464.

The following results are proved. If ρ, τ and s are the radius of curvature, the torsion, and the arc length, of a closed curve $\mathfrak{L}$ on a sphere, and n is any integer (positive or negative), then the integral

$$I_n = \oint_{\mathfrak{L}} \rho^n \tau ds$$

round the closed curve has the value zero. Conversely if $I_n = 0$ for some fixed n round every closed curve on a surface F, then F is a sphere. *P. Du Val* (London)

5968:

Bernhart, Arthur. Curves of general pursuit. Scripta Math. **24** (1959), 189–206.

Continuation of historical and expository account.

5969:

Švec, Alois. Les surfaces *R* dans les espaces projectifs de dimension impaire. Czechoslovak Math. J. **9 (84)** (1959), 243–264. (Russian summary)

A study of surfaces, in a projective space S_{2n+1}, admitting a conjugate net; in particular, of the surfaces R such that their quasi-asymptotic curves correspond to the quasi-asymptotic curves on the neighboring Laplace transforms. All the surfaces of the Laplace sequence of an R-surface are R-surfaces.

Two surfaces are said to be in a C_s deformation if it is possible to establish between them a correspondence such that the neighborhoods of order s of corresponding points correspond in a collineation. It is proved that the only surfaces which admit C_{n+1} deformations are R-surfaces. The last section of the paper is devoted to the study of congruences of lines in S_{2n+1}. *E. Bompiani* (Rome)

5970:

Vaccaro, Giuseppe. Topologia differenziale delle calotte tridimensionali tangenti in un punto. Ann. Mat. Pura Appl. (4) **44** (1957), 201–231.

This paper is concerned with the differential topology of caps $\sigma_3{}^2$ (a cap $\sigma_k{}^s$ is the set of all differentiable manifolds $X_k \subset X_n$, $k<n$, having a contact of order s at a point O, center of the cap); namely, two tangent caps $\sigma_3{}^2$ and their invariant properties with respect to all point-transformations regular at the self-corresponding point which is their common center are investigated; such a pair of caps is indicated by $(\underset{1}{\sigma}, \underset{2}{\sigma})$. The main problem is to find the minimum dimensionality κ of a manifold V_κ containing the $(\underset{1}{\sigma}, \underset{2}{\sigma})$: κ may vary from 4 to 9. To examine the different possibilities it is necessary to consider first the principal 2-plane $\sigma_2{}^1$ determined by two tangent curvilinear differential elements of order 2, E^2, through O and with the same tangent belonging to $\underset{1}{\sigma}$ and $\underset{2}{\sigma}$; then the locus of these principal $\sigma_2{}^1$ for a given tangent (varying the E^2's) or (2, 1)-osculating cap; then the loci of these osculating caps varying the tangent in a 2-plane $\sigma_2{}^1$ or in all possible ways. Already these loci may give rise to different configurations which may be properly described in projective spaces. All possibilities are carefully analyzed for $\kappa = 4$, 5 and 9. It is to be noticed that even the facts relating to a single cap $\sigma_3{}^2$ in a projective space were not fully known. *E. Bompiani* (Rome)

5971:

Longo, Carmelo. Calotte regolari tridimensionali del secondo ordine. Ann. Mat. Pura Appl. (4) **46** (1958), 369–398.

The author restates the definition of a space cap $\sigma_3{}^2$ and describes the various spaces associated with a $\sigma_3{}^2$. He has already studied projectively the $\sigma_3{}^2$ for which $S(2) = S_9$ [Univ. Roma Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **9** (1950), 280–292; MR **16**, 282], and G. Vaccaro has studied the cases $S(2) = S_h$, $h = 4$ and 5 [review above]. Here different projective characterizations are presented for the cases $h = 4$ and 5, the cases $h = 6$, 7, 8 are also characterized projectively, and projective properties of the various $\sigma_3{}^2$ are deduced.

With a $\sigma_3{}^2$ for which $S(2) = S_{9-h_1}$ ($h_1 \geq 1$) there is associated a linear system of ∞^{h_1-1} characteristic cones and the problem of the characterization of the $\sigma_3{}^2$ is carried back to that of the cones. Two theorems are basic in this work, one stating a necessary and sufficient condition that a $\sigma_3{}^2$ have a characteristic cone and the other that it have a degenerate characteristic cone.

The author also reviews the topological characterization of pairs of tangent $\sigma_3{}^2$'s, shows how the projective study of one $\sigma_3{}^2$ and the topological study of a pair of tangent $\sigma_3{}^2$'s differ for the case $h = 9$, but how his work on the projective case can be carried over to the topological case for $h \leq 8$. *A. Schwartz* (New York, N.Y.)

5972:

★Trajdos-Wróbel, Tadeusz. Wstęp do analizy wektorowej. [Introduction to vector analysis.] Państwowe Wydawnictwo Naukowe, Warsaw, 1959. 196 pp. zł. 17.00.

Vector algebra, vector analysis (theorems of Ostrogradski, Stokes and Green), and a short introduction to tensor analysis.

5973:

Nevanlinna, Rolf. Über Tensorrechnung. Rend. Circ. Mat. Palermo (2) **7** (1958), 285–302.

This paper presents an "indexfree" formulation of many aspects of tensor calculus. As an example of the notation: $\overset{\beta}{\underset{\alpha}{T}}$ represents a tensor of β contravariant indices

and α covariant indices. The ideas are illustrated in connection with contraction, exterior differentiation, covariant differentiation, and similar topics.

C. B. Allendoerfer (Seattle, Wash.)

5974:

Fulton, Curtis M. Quasi vectors. Tensor (N.S.) **9** (1959), 44–46.

By means of the generalized Kronecker deltas and the differential derivatives the author constructs $\Delta_{\alpha'}{}^{a} = \delta_i{}^{a} A_{\alpha'}{}^{i}$ and defines a quasivector by the transformation $V^a = \Delta_{\alpha'}{}^{a} V^{\alpha'}$. Identifications with ordinary alternating tensors and some relations for ϵ-systems are added.

E. M. Bruins (Amsterdam)

5975:

Singh, H. D. The second extension of a covariant differentiation process. Riv. Mat. Univ. Parma **8** (1957), 397–404.

The author generalizes the process of covariant differentiation of tensors whose components are functions not only of the coordinates x^i but also of their m derivatives $x^{(1)i}, x^{(2)i}, \cdots, x^{(m)i}$. The results obtained by the present author contain as special cases the results obtained by H. V. Craig [Bull. Amer. Math. Soc. **37** (1931), 731–734], Marie M. Johnson [Bull. Amer. Math. Soc. **46** (1940), 269–271; MR **1**, 273] and the author himself [Tensor **7** (1957), 137–140; MR **20** #2738].

K. Yano (Hong Kong)

5976:

Kuczma, Marek. On linear differential geometric objects of the first class with one component. Publ. Math. Debrecen **6** (1959), 72–78.

Every linear differential geometric object of the first class with one component must be of one of the two forms,

$$x' = x + \ln|\varphi(\Delta)| \quad \text{or} \quad x' = \varphi(\Delta)x + c[\varphi(\Delta) - 1],$$

where Δ is the Jacobian of the transformation, c any constant, and φ a function satisfying $\varphi(u)\varphi(v) = \varphi(uv)$ for $u, v \neq 0$. Every quasi-linear (i.e., equivalent to a linear) differential object of the first class is equivalent to the object $x' = \varphi(\Delta)x$, with φ as before. The measurable φ lead to the density $x' = \Delta x$, the Weyl density $x' = |\Delta|x$ and the bi-scalar $x' = \operatorname{sgn}\Delta \cdot x$. The proofs depend heavily on a forthcoming result of M. Kucharzewski classifying real-valued functions of matrices with $f(AB) = f(A)f(B)$ as $f(A) = \varphi(\det A)$, φ as before.

A. Nijenhuis (Seattle, Wash.)

5977:

Laugwitz, Detlef. Beiträge zur affinen Flächentheorie mit Anwendungen auf die allgemein-metrische Differentialgeometrie. Bayer. Akad. Wiss. Math.-Nat. Kl. Abh. **93** (1959), 59 pp.

The theory of hypersurfaces is studied in a centro-affine space, i.e., in a space with the group of centro-affine transformations as its group of automorphisms, by a method essentially different from that of E. Salkowski [*Affine Differentialgeometrie*, de Gruyter, Berlin-Leipzig, 1934] and O. Mayer [Ann. Sci. Univ. Jassy **21** (1935), 1–77]. This space is essentially nothing but a vector space, and a hypersurface appears as an indicatrix (i.e., closed convex hypersurface) in a tangent space of the differentiable manifold with a general metric (for example, Finsler and Cartan manifolds). For this reason a large part of this paper is devoted to the theory of closed convex hypersurfaces. Chapter I deals with the osculating quadrics of the hypersurface in detail and especially with its sections by hyperplanes parallel to its tangent hyperplane, where some new relations to the equivolume-affine differential geometry are found. Although several special osculating quadrics, e.g., Darboux's and Lie's quadrics, are known to play important roles in the affine geometry, it is shown in chapter II that the whole affine geometry of hypersurfaces can be found on osculating quadrics not only for the centro-affine geometry but also for the equivolume-affine geometry. That is, the fundamental forms of the hypersurface are obtained from the osculating quadrics geometrically. In chapter III there are applications to the general metric differential geometry, and some affine-geometrical meanings are given in the theory of general metric spaces. At the end the author deals with the space-problem by the same method.

A. Kawaguchi (Sapporo)

5978:

Ryžkov, V. V. Conjugate systems on multidimensional spaces. Trudy Moskov. Mat. Obšč. **7** (1958), 179–226. (Russian)

This is a systematic investigation of the general case, in which on a hypersurface $\bar{x} = \bar{x}(u^i)$, $i = 1, \cdots, n$ of an affine E_N there are conjugate directions, especially when these directions are holonomous. Here we call two directions δ_1 and δ_2, $\delta = a^i\partial/\partial u^i$, conjugate if $\delta_1\delta_2\bar{x} = \delta_2\delta_1\bar{x} \equiv 0(E_n)$ where $\bar{a} \equiv 0(E_n)$ means that $\bar{a}$ lies in the tangent plane of $\bar{x}$. Similarly, two directions E_p and E_q on $\bar{x}$ are conjugate if this relation holds for all directions of E_p in relation to all directions on E_q: $d_{E_p}d_{E_q}\bar{x} \equiv 0(E_n)$. A particular case is that of asymptotic E_p, E_q, if for all E_s in which E_p and E_q intersect $d_{E_s}{}^2\bar{x} \equiv 0(E_n)$. The general case considered is that in which a conjugate system $S(E_{p_1}, E_{p_2}, \cdots, E_{p_r})$, $p_i > 0$, $r \geqq 2$, exists such that the E are pairwise conjugate and isolated (that is, the dimension of E_{p_1} and E_{p_2} together is $p_1 + p_2$, etc.). In particular, the system $S(E_{p_0}{}^a, E_{p_1}, \cdots, E_{p_r})$ is called complete, if $2p_0 + p_1 + \cdots + p_r \geqq n$; here $E_{p_0}{}^a$ is asymptotic. A special study is devoted to n-conjugate systems (here through each point pass n pairwise-conjugate directions) and to completely stratifiable conjugate systems. [In the bibliography, special reference is paid to a paper by P. O. Bell, Duke Math. J. **21** (1954), 323–327; MR **16**, 168].

D. J. Struik (Cambridge, Mass.)

5979:

Su, Buchin. On Demoulin transforms of projective minimal surfaces. I. Acta Math. Sinica **7** (1957), 28–50. (Chinese. English summary)

A translation of this article is reviewed below.

5980:

Su, Buchin. On Demoulin transforms of projective minimal surfaces. I. Sci. Sinica **6** (1957), 941–965.

G. Thomsen hat gezeigt, daß eine projektive Minimalfläche S und eine ihrer 4 Demoulin-Transformierten $\hat{S}$ in Asymptotenkorrespondenz stehen. Verf. zeigt in der vorliegenden Arbeit: (a) Die Asymptotenkorrespondenz

besteht zu allen 4 D-Transformierten von S; (b) mit S sind alle 4 D-Transformierten projektive Minimalflächen. Daraufhin untersucht Verf. die zweiten D-Transformierten von S. Davon gibt es 9; eine ist die Ausgangsfläche S, weitere 4 sind W-Transformierte von S. Es werden dann noch die Godeaux-Ketten von S und einer seiner D-Transformierten betrachtet. Bei Übertragung auf die Plückerquadrik ergeben sich dort 2 räumliche Streckenzüge, deren Geraden sich gegenseitig in bestimmter Weise schneiden. *W. Burau* (Hamburg)

5981:

Su, Buchin. On Demoulin transforms of projective minimal surfaces. III. Acta Math. Sinica 7 (1957), 123–127. (Chinese. English summary)

[Editor's note: Part II is Fuh-tan J. Nat. Sci. Cl. 1 (1956), 111–119.] Let S be a projective minimal surface and $\hat{S}$ one of its Demoulin transforms. Let the Godeaux sequences of S and $\hat{S}$ be

$$\cdots, U_n, \cdots, U_1, U, V, V_1, \cdots, V_m, \cdots,$$
$$\cdots, \hat{U}_n, \cdots, \hat{U}_1, \hat{U}, \hat{V}, \hat{V}_1, \cdots, \hat{V}_m, \cdots,$$

respectively. The following theorem is proved. If the corresponding points of the Godeaux sequences of S and $\hat{S}$ in a 5-dimensional linear space S_5 be arranged in three rows:

$$\begin{array}{cccccccccc} \cdots U_{n+1} & U_n & \cdots & U_3 & U_2 & U_1 & U & \cdots & V_m & V_{m+1} \cdots, \\ \cdots \hat{U}_{n-1} & \hat{U}_{n-2} & \cdots & \hat{U}_1 & \hat{U} & \hat{V} & \hat{V}_1 & \cdots & \hat{V}_{m+2} & \hat{V}_{m+3} \cdots, \\ \cdots U_{n-3} & U_{n-4} & \cdots & V & V_1 & V_2 & V_3 & \cdots & V_{m+4} & V_{m+5} \cdots, \end{array}$$

then the join of any two consecutive points in the middle row must intersect the join of the two consecutive points standing in the same columns of the first or third row, and therefore the corresponding points of the Godeaux sequences constitute two broken lines, generally extended in both directions, which intersect each other at intervals of three sides. *T. K. Pan* (Norman, Okla.)

5982:

Su, Buchin. On Demoulin transforms of projective minimal surfaces. IV. Acta Math. Sinica 8 (1958), 239–242. (Chinese. English summary)

This paper supplements the result in III [above]. It considers two rectilinear congruences W associated with S and $\hat{S}$ in S_5 and demonstrates several relations between the Laplace sequences of W and the Godeaux sequences of S and $\hat{S}$. *T. K. Pan* (Norman, Okla.)

5983:

Su, Buchin. On Demoulin transforms of projective minimal surfaces. V. Acta Math. Sinica 8 (1958), 276–280. (Chinese. English summary)

This is a sequel to the previous paper. *T. K. Pan* (Norman, Okla.)

5984:

Su, Buchin. Contributions to the theory of projective minimal surfaces. Rev. Math. Pures Appl. 3 (1958), 173–189.

This paper consists of further results in addition to those obtained in a series of papers by the author on a projective minimal surface S and its Demoulin transforms [An. Sti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **2** (1956), 61–67; MR **20** #3562; Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 569–576; MR **19**, 1075; and #5979–5983 above]. Let $\bar{S}$, $S_{(1)}$ and $S_{(2)}$ be Demoulin transforms of S such that $S_{(1)}$ and $S_{(2)}$ are also transforms W of S. Let (J) and $(\bar{J})$ be the Laplace sequences corresponding to W transforming S to $S_{(1)}$ and $S_{(2)}$ respectively. Let $\{\phi_n\}$, $\{\bar{\phi}_n\}$, $\{\phi_n'\}$ and $\{\phi_n''\}$ be the Godeaux sequences of quadrics of S, $\bar{S}$, $S_{(1)}$ and $S_{(2)}$ respectively. Some remarkable entities associated with (J) and $(\bar{J})$ are considered and relations between $\{\phi_n\}$, $\{\bar{\phi}_n\}$, $\{\phi_n'\}$ and $\{\phi_n''\}$ at corresponding points are discussed. *T. K. Pan* (Norman, Okla.)

5985:

Su, Buchin. A remarkable class of surfaces in projective space. Lucrăr. Şti. Inst. Ped. Timişoara. Mat.-Fiz. **1958**, 65–72 (1959). (Romanian. English and Russian summaries)

A surface of the class of surfaces for which the diagonals of Demoulin quadrilaterals belong to a linear congruence is called a surface of Bol. With the help of certain formulae given in #5984, the following theorem is proved. A projective minimal surface is a surface of Bol when, and only when, its associate sequence (G) of Godeaux is a periodic sequence of period 4. *T. K. Pan* (Norman, Okla.)

5986:

Su, Buchin. A problem in the projective theory of surfaces. Sci. Record (N.S.) **3** (1959), 143–148.

Der Verfasser stellt sich die Aufgabe, alle projektiven Minimalflächen anzugeben, deren assoziierte Laplacekette (L) periodisch ist. Dies Problem läßt sich im allgemeinen auf das folgende reduzieren: Alle Flächen mit periodischer Godeaux-Kette (G) anzugeben. Bisher wußte man nur folgendes: Wenn S eine Bol-Fläche ist, d.h., eine solche, deren Demoulin-Diagonalen einer linearen Kongruenz angehören, dann hat die zu S gehörige Kette (L) die Periode 4, aber die Kette (G) ist nicht periodisch; allgemein ist die Periode von (L) stets gerade. Verf. zeigt nun: Bei $p \neq 4$ besitzen beide obigen Folgen die gleiche Periode, und die Bolschen Flächen sind auch durch $p=4$ gekennzeichnet. *W. Burau* (Hamburg)

5987:

Rozenfel'd, B. A. Quasi-elliptic spaces. Trudy Moskov. Mat. Obšč. **8** (1959), 49–70. (Russian)

When in a projective space P_n with coordinates $x^0, x^1, \cdots, x^n$ an "absolute cone" $\sum_a (x^a)^2 = 0$ $(a, b = 0, 1, \cdots, m)$ with "vertex" $x^0 = x^1 = \cdots = x^m = 0$ is given, and an "absolute quadric" $\sum_u (x^u)^2 = 0$ $(u, v = m+1, \cdots, n)$ in this "vertex", then the P_n becomes a "quasi-elliptic space", denoted by R_n^m. The "vertex" $x^a = 0$ is called the absolute plane. For $m=0$ we obtain the euclidean R_n. The R_3^1 has appeared in Blaschke and Müller's *Ebene Kinematik*, Oldenbourg, München, 1956 [MR **17**, 1245], I. I. Železina, Dokl. Akad. Nauk SSSR **106** (1956), 959–962 [MR **17**, 996] and E. A. Hatipov, Trudy Sem. Vektor Tenzor. Anal. **10** (1956), 285–310 [MR **18**, 820]. Similarly,

by singling out in P_n an $(n-m-1)$-dimensional hyperplane as "improper plane" a "quasi-affine" space A_n^m is obtained; for $m=0$ we have the ordinary affine A_n. If $x^a=0$, $a=0, 1, \cdots, m$, represents the improper plane, then "quasi-affine transformations" take the form $'x^a=\sum A_b{}^a x^b$, $'x^u=\sum C_b{}^u x^b+\sum B_v{}^u x^v$ with the matrices A and B non-singular, of order $m+1$ and $n-m$ respectively. An R_n^m can be understood as an A_n^m with a metric. An m-dimensional plane of A_n^m which does not intersect the improper plane can be given by a quadratic matrix $X_0=(x_a{}^b)$ and a rectangular matrix $X_1=(x_a{}^u)$. It is now shown how the geometry of such planes can be developed as a matrix calculus, and how their quasi-affine invariants can be found. Then follow sections on quasi-elliptic motions and metrical invariants of planes and quadrics. For this method of rectangular matrices is cited a paper, "Metrical invariants of quadrics in quasi-elliptic spaces", by L. P. Pticyna, L. V. Pučkova and L. V. Rumyanceva, but the journal reference seems incorrect.

D. J. Struik (Cambridge, Mass.)

5988:

Pinl, M. Zur Differentialgeometrie im totalisotropen R_2 und R_3 eines komplexen euklidischen R_4 und R_6. Monatsh. Math. **63** (1959), 256–264.

This paper is an elaboration of the fact [K. Strubecker, Deutsch. Akad. Wiss. Berlin Schr. Forschungsinst. Math. **1** (1957), 143–155; MR **19**, 312] that the equiformal differential geometry of a totally isotropic R_n in complex euclidean R_{2n} is essentially the same as the affine differential geometry of an affine space. Following the pattern of affine differential geometry (using some fixed auxiliary vectors, say $\mathfrak{v}$, $\mathfrak{w}$, to form determinants like $(\mathfrak{x}', \mathfrak{x}'', \mathfrak{v}, \mathfrak{w})$), concepts such as quasi-arc length and quasi-curvature of curves are defined and examined. Similarly, the theory of surfaces is studied.

A. Nijenhuis (Seattle, Wash.)

5989:

Hwang, Cheng-chung. Miscellaneous notes on Riemannian geometry. Tensor (N.S.) **9** (1959), 99–103.

The author proves the following three theorems. (1) Let an $(n+1)$-dimensional Riemannian space V_{n+1} admit a hypersurface V_n with indeterminate lines of curvature; then the normal to V_n at a point of V_n is a Ricci principal direction for V_{n+1} if and only if the mean curvature Ω of hypersurface is constant. The if-part of this theorem is found in Eisenhart's *Riemannian geometry*, p. 181 [Princeton Univ. Press, 1949; MR **11**, 687]. (2) A 2-dimensional surface admits a concircular correspondence with another one, if and only if it is isometric with a surface of revolution. The n-dimensional case was dealt with by the reviewer [Proc. Imp. Acad. Tokyo **16** (1940), 195–200, 354–360; MR **2**, 165]. (3) If an m-dimensional affinely connected space V_m without torsion contains $m+2$ families of totally geodesic hypersurfaces such that there exists a mapping of V_m into an affine space A_m which brings the $m+2$ families of hypersurfaces onto $m+2$ families of planes in general position, then the space V_m must be projectively flat, and conversely. [Hu Hoo-sung, Acta Math. Sinica **8** (1958), 269–271; MR **20** #6064.] {Reviewer's Note: To prove (1) the author takes a special coordinate system, but it seems to the reviewer that we do not have to take such a coordinate system. In this case, the Weingarten equation is $\nabla_c N^\kappa=-\Omega B_c{}^\kappa$; thus from the Ricci formula, we get

$$B_c{}^\nu B_b{}^\mu N^\lambda K_{\nu\mu\lambda}{}^\kappa = -[(\nabla_c\Omega)B_b{}^\kappa-(\nabla_b\Omega)B_c{}^\kappa],$$

and hence

$$K_\lambda{}^\kappa N^\lambda = (K_{\mu\lambda}N^\mu N^\lambda)N^\kappa+(n-1)'g^{cb}(\nabla_c\Omega)B_b{}^\kappa,$$

which proves the theorem.}

K. Yano (Hong Kong)

5990:

Feeman, George F.; and Hsiung, Chuan-Chih. Characterizations of Riemann n-spheres. Amer. J. Math. **81** (1959), 691–708.

Let M_α be the αth mean curvature of a closed orientable hypersurface V^n $(n\geqq 2)$ of class C^3 imbedded in the Riemannian manifold V^{n+1} such that "there is a normal coordinate system of Riemann at a fixed point O covering the whole manifold V^{n+1}". Then some characterizations of Riemann n-spheres of the following type are given: Suppose that there exists an odd integer s $(1<s\leqq n)$ such that at all points of the hypersurface V^n the function p (=scalar product of the unit normal vector of V^n at a point P and the position vector of P with respect to O) is of the same sign, $M_i>0$ for $i=1, 2, \cdots, s-1$ and either M_{s-1} or M_s is constant; then V^n is a Riemann n-sphere. The case where V^{n+1} is of constant curvature is first considered. [For analogous, though less general results, see Hsiung, Math. Scand. **2** (1954), 286–294; Pacific J. Math. **6** (1956), 291–299; MR **16**, 849; **18**, 507.]

L. A. Santaló (Buenos Aires)

5991:

Mihăilescu, Tiberiu. La courbure extérieure des hypersurfaces non holonomes. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **1** (49) (1957), 435–448.

The object is to generalize the notion of Gaussian curvature to a non-holonomic hypersurface $V^{n-1}{}_n$ of a general Riemannian space V_n. Using the classical method of spherical representation as a guide, the author defines a scalar Gaussian curvature at points of a non-holonomic hypersurface $V^{n-1}{}_n$. This leads to the notion of vector Gaussian curvature, i.e., exterior curvature, and it is shown that this concept is useful when discussing problems of parallel transport.

T. J. Willmore (Liverpool)

5992:

Narasimhan, M. S. A remark on curvature and the Dirichlet problem. Bull. Amer. Math. Soc. **65** (1959), 363–364.

Let M be an oriented C^∞ Riemannian manifold (countable at infinity), with mean curvature positive and bounded away from zero. Let Δ be the Friedrichs extension of the Laplacian on compact carried one-forms on M. The curvature condition immediately implies the Poincaré inequality for compact carried C^∞ one-forms, from which it follows that there are no non-zero one-forms in the harmonic space of Δ. From a previous work of the author [J. Indian Math. Soc. **20** (1956), 291–297; MR **19**, 980] this implies for one-forms on M that the "Dirichlet problem" is solvable and that there is a Green's form for Δ.

M. P. Gaffney (Washington, D.C.)

5993:

Sumitomo, Takeshi. Projective and conformal transformations in compact Riemannian manifolds. Tensor (N.S.) **9** (1959), 113–135.

Since S. Bochner [Bull. Amer. Math. Soc. **52** (1946), 776–797; MR 8, 230] proved the non-existence of a one-parameter group of motions in a compact Riemannian space M with negative definite Ricci curvature, there have appeared many papers on the existence and non-existence of motions and affine collineations in compact or non-compact Riemannian spaces. The main purpose of the present paper is to discuss the existence and non-existence of projective and conformal transformations in compact symmetric Riemannian spaces, or a little bit more generally, in compact Ricci symmetric Riemannian spaces, a Ricci symmetric Riemannian space being a Riemannian space in which the Ricci tensor has vanishing covariant derivative.

The author denotes by $I(M)$, $A(M)$, $P(M)$, $H(M)$ and $C(M)$ the totality of isometric, affine, projective, homothetic and conformal transformations in M, respectively, and by $I_0(M)$, $A_0(M)$, $P_0(M)$, $H_0(M)$ and $C_0(M)$ their connected components of the identical transformation, respectively.

After having stated these definitions in § 1, the author enumerates known results as lemmas in § 2. In § 3, the author studies projective transformations in symmetric or Ricci symmetric Riemannian spaces and obtains the following theorems. (1) Symmetric Riemannian spaces are classified into (i) spaces of constant sectional curvature, (ii) spaces admitting no proper projective transformation. (2) In a compact Ricci symmetric Riemannian space, at least one of the following conditions should be satisfied. (i) The space is an Einstein space. (ii) $P_0(M)$ consists of motions. (3) A Ricci symmetric Riemannian space can be classified into the following two categories. (i) The space is an Einstein space. (ii) $P_0(M)$ consists of affine transformations.

In § 4 the author studies conformal transformations in symmetric or Ricci symmetric Riemannian spaces and obtains the following theorems. (4) In a locally homogeneous Riemannian space, one of the following two conditions should be satisfied. (i) The space is locally Euclidean. (ii) Proper homothetic transformations cannot exist. (5) In a Ricci symmetric Riemannian space, one of the following two conditions should be satisfied. (i) The space has vanishing Ricci tensor. (ii) Proper homothetic transformations cannot exist. (6) In a locally homogeneous Riemannian space, one of the following two conditions should be satisfied. (i) The space is conformally Euclidean. (ii) Proper conformal transformations cannot exist. (7) In a compact Ricci symmetric Riemannian space, one of the following conditions should be satisfied. (i) The space is an Einstein space with positive curvature scalar. (ii) $C_0(M)$ consists of motions. (8) Compact symmetric spaces are classified into the following two spaces. (i) The space is of positive constant sectional curvature; (ii) $C_0(M)=I_0(M)$ and the space is neither Einstein space nor conformally flat. (9) In a compact Ricci symmetric Riemannian space, one of the following two cases can occur. (i) The space is of positive constant sectional curvature. (ii) $C_0(M)=I_0(M)$. (10) Ricci symmetric spaces are classified into two spaces satisfying (i) $n(n-1)R_j^iR_k^jR_i^k-(2n-1)RR^{ij}R_{ij}+R^3=0$; (ii) $C(M)=I(M)$.

In the last paragraph the author studies characteristic roots of the Ricci tensor and the existence of proper conformal transformations in symmetric Riemannian spaces.

K. Yano (Hong Kong)

5994:

Takasu, Tsurusaburo. Extended Euclidean geometry and extended equiform geometry under the extensions of respective transformation groups. I. Yokohama Math. J. **6** (1958), 89–176.

In a Euclidean space E^n with rectangular cartesian coordinates (x^n) the n Pfaffians $\omega^l=\omega_m{}^l dx^m$ are given $(l, m, \cdots=1, 2, \cdots, n)$, defining $ds^2=\omega^l\omega^l$. The solution to the extremal problem $\delta s=0$ in the manifold $(\omega_m{}^l, x^n)$ is given by $(d/ds)(\omega^l/ds)=0$, its integral is written

$$\xi^l = \int\left(\frac{\omega^l}{ds}\right)ds = a^l s+c^l.$$

The curves thus defined are called II-geodesic curves corresponding to the $\omega_m{}^l$, along them $dx^l/ds=a^p\Omega_p{}^l$, where $\Omega_l{}^p\omega_q{}^l=\delta_q{}^p$. The (ξ^l) are called II-geodesic coordinates. If the integral of $d\xi^l=\omega_m{}^l dx^m$ is written $\xi^l=a_m{}^l x^m+a_0{}^l$, then the $a_m{}^l$ are functions of the x^n; $a_0{}^l$ are constants. The group $\mathfrak{G}$ consisting of $(a_m{}^l, a_0{}^l)$ is called the extended Euclidean group, it has the classical Euclidean group $\mathfrak{E}$ as subgroup. If $\mathfrak{G}=\mathfrak{E}\mathfrak{H}=\mathfrak{H}\mathfrak{E}$, and $\mathfrak{H}=h_0+h_1+h_2+\cdots$ $(h_0=1)$, then $\mathfrak{G}=\mathfrak{E}h_0+\mathfrak{E}h_1+\mathfrak{E}h_2+\cdots$. Each transformation $h_i=(a_m{}^l, a_0{}^l)$ transforms the space $E^n=E_{h_0}{}^n$ into another $E_{h_i}{}^n$, and the whole space to be considered is $E_{\mathfrak{H}}{}^n=E_{h_0}{}^n+E_{h_1}{}^n+\cdots$. The geometry belonging to $\mathfrak{G}$ is the extended Euclidean geometry (extended by $\mathfrak{H}$), and the paper gives a detailed study of its trigonometry, theory of curves and surfaces, including triply orthogonal systems.

D. J. Struik (Cambridge, Mass.)

5995:

Mihăilescu, Tiberiu. Théorèmes projectifs du type Gauss-Bonnet. Rev. Math. Pures Appl. **3** (1958), 63–86.

A projective theorem of Gauss-Bonnet type on a surface S is a relationship of the form $\int_C \Omega=\iint_\Delta D\Omega$, where Δ is a simply connected region of S, C is its boundary, and D represents exterior differentiation. The form Ω is a linear differential form defined in an invariant manner and which is also a projective invariant. The purpose of this paper is to find such theorems.

The paper contains a rather full exposition of the basic formulas of the projective geometry of S and then proceeds to classify invariant forms according to the differential order of their coefficients. Forms of orders 3, 4, and 5 are obtained. A form Ω is called "stable" if $D\Omega=0$. For each Ω there are certain surfaces S on which it is stable, and these are described explicitly. A form which exists in general may fail to be defined on certain other surfaces and these are also described. Geometric interpretations are given throughout and exceptional surfaces are frequently identified with the mathematicians who had previously studied them.

C. B. Allendoerfer (Seattle, Wash.)

5996:

Elianu, Jean. Sur certains espaces riemanniens. Rev. Math. Pures Appl. **3** (1958), 389–394.

Let V be an orientable Riemannian manifold of dimension $4m$. A current T of degree $2m$ is called

self-adjoint if $*T=T$. The author shows that in general a current T is the sum $T=T_a+T_0$ where T is self-adjoint and $*T_0=-T_0$. He obtains a similar decomposition for harmonic forms in case V is compact, and gives necessary and sufficient conditions for a harmonic form to be self-adjoint. *J. J. Kohn* (Waltham, Mass.)

5997:

Ahn, Jae Koo. On general harmonic and Killing vectors in metric manifolds with general symmetric affine connection. Kyungpook Math. J. **2** (1959), 23–31.

Étude des champs harmoniques et des vecteurs de Killing sur une variété métrique pourvue d'une connexion symétrique affine générale (non liée à la métrique). Les résultats sont analogues à ceux obtenus par K. Yano et S. Bochner [*Curvature and Betti numbers*, Princeton Univ. Press, 1953; MR **15**, 989; Ch. VII]. *J. Lelong* (Paris)

5998:

Kashiwabara, Shōbin. The decomposition of a differentiable manifold and its applications. Tôhoku Math. J. (2) **11** (1959), 43–53.

Theorems on the global decomposition of Riemannian manifolds have been given by de Rham for simply connected manifolds [Comment. Math. Helv. **26** (1952), 328–344; MR **14**, #584], and by the reviewer under more general conditions [Proc. London Math. Soc. (3) **3** (1953), 1–19; MR **15**, 159]. Analogous theorems are now given for manifolds with affine connection which are locally decomposable in a well defined sense, and also for locally decomposable Finsler manifolds.

A. G. Walker (Seattle, Wash.)

5999:

Vranceanu, G. Tenseurs harmoniques et groupes de mouvement d'un espace de Riemann. Comment. Math. Helv. **33** (1959), 161–173.

Soit $\{\eta_a\}$ une algèbre de Lie de vecteurs tangents à une variété riemannienne V^n; à chaque tenseur antisymétrique d'ordre p, soit $\xi_{i_1\ldots i_p}$, l'A. fait correspondre les tenseurs antisymétriques d'ordre $p-q$ définis par

$$\xi_{i_1\ldots i_{p-q}a_1\ldots a_q}=\eta_{a_1}{}^{j_1}\cdots\eta_{a_q}{}^{j_q}\xi_{i_1\ldots i_{p-q}j_1\ldots j_q}$$

et il étudie l'invariance de ces tenseurs dans les transformations infinitésimales définies par les vecteurs η_a.

Ces résultats sont utilisés dans la deuxième partie pour la recherche des formes différentielles fermées invariantes dans un groupe transitif d'isométries. L'A. en déduit divers résultats relatifs aux nombres de Betti de la variété, généralisant ceux relatifs aux variétés de groupes [Cf. W. V. D. Hodge, *Harmonic integrals*, 2nd ed., University Press, Cambridge, 1952; MR **14**, 500]. *J. Lelong* (Paris)

6000:

Yano, Kentaro. Affine connexions in an almost product space. Kōdai Math. Sem. Rep. **11** (1959), 1–24.

Following results on almost-product structures obtained by A. G. Walker [Quart. J. Math. Oxford Ser. (2) **6** (1955), 301–308; **9** (1958), 221–231; MR **19**, 312; **20** #6135] and T. J. Willmore [ibid. **7** (1956), 269–276; Proc. London Math. Soc. (3) **6** (1956), 191–204; MR **20** #4299; **19**, 455], the author derives or re-derives 25 theorems, mainly relations between an almost-product structure and affine connections adapted to it in various ways. Extensive use is made of the methods of the "rigged $X_n{}^m$ in X_n and L_n" of chapter V, § 7 of J. A. Schouten's *Ricci calculus* [Springer, Berlin, 1954; MR **16**, 521], and many explicit formulas are obtained. *A. Nijenhuis* (Seattle, Wash.)

6001:

Wang, Hsien-Chung. On invariant connections over a principal fibre bundle. Nagoya Math. J. **13** (1958), 1–19.

The purpose of the paper is to discuss the connections over a principal fibre bundle which admits a fibre-transitive Lie group of automorphisms.

Let $\{E,S\}$ be a differentiable principal fibre bundle with total space E, structural group S and base space B. Let G be a Lie group of automorphisms of $\{E,S\}$ and J the subgroup leaving a fibre F_0 invariant. There is a natural homomorphism $\psi: J\to S$. If we regard G as a transformation group of the base space B, then J is the isotropic subgroup at the point $b_0\in B$ which corresponds to F_0. Denote by $\hat{\mathfrak{G}}$, $\hat{\mathfrak{S}}$, $\hat{\mathfrak{J}}$ the Lie algebras of G, S, J respectively.

After discussions on φ-invariant differential forms, automorphisms of a principal fibre space and invariant connections, the author proves his first theorem: (1) Let G be a Lie group of automorphisms of a differentiable fibre space $\{E,S\}$, transitive on the fibres. Then there is a one-to-one correspondence between the invariant connections ω over E and the linear mappings $\Psi:\hat{\mathfrak{G}}\to\hat{\mathfrak{S}}$, such that

$$\Psi\circ(\mathrm{Ad}\,j)=\mathrm{Ad}\,\psi(j)\circ\Psi,\quad \Psi(\mathfrak{j})=\psi(\mathfrak{j})\quad (j\in J,\ \mathfrak{j}\in\hat{\mathfrak{J}}).$$

Let Ω denote the curvature form of ω. Then the restriction of $p^*\Omega$ on G is given by

$$2(p^*\Omega)(\mathfrak{g}_1,\mathfrak{g}_2)=[\Psi(\mathfrak{g}_1),\Psi(\mathfrak{g}_2)]-\Psi[\mathfrak{g}_1,\mathfrak{g}_2]\quad(\mathfrak{g}_1,\mathfrak{g}_2\in\hat{\mathfrak{G}}).$$

He then defines the group Δ. For each point x of E, we denote by Σ_x the holonomy group at x and by $\mathfrak{H}_x$ the set of all points which can be joined to x by horizontal curves. Since elements of both G and S carry horizontal curves into horizontal curves, we have

$$g(\mathfrak{H}_x)=\mathfrak{H}_{g(x)},\quad s(\mathfrak{H}_x)=\mathfrak{H}_{s(x)}\quad(g\in G,\ s\in S).$$

Now fix a point x of E and consider the orbit $\mathfrak{E}_x=G(\mathfrak{H}_x)$ of $\mathfrak{H}_x$ under G. For each u of $\mathfrak{E}_x$, we put

$$\Delta_u=\{s:s\in S,\ s(u)\in\mathfrak{E}_x\}.$$

Then we see that Δ_u forms a group and is independent of the choice of u in $\mathfrak{E}_x$. Nevertheless, if x is changed Δ_u is changed into one of its conjugate subgroups in S.

Then the author proves: (2) Let G be a Lie group of automorphisms of $\{E,S\}$, transitive on the fibres. Suppose that ω is an invariant connection corresponding to the linear mapping $\Psi:\hat{\mathfrak{G}}\to\hat{\mathfrak{S}}$. Denote by $\hat{\mathfrak{R}}$ the subalgebra of $\hat{\mathfrak{S}}$ generated by $\Psi(\hat{\mathfrak{G}})$. Then $\Delta_{x_0}=\psi(J)\cdot R$ where $R=\exp\hat{\mathfrak{R}}$. If, moreover, $\psi(J)$ is arcwise connected, then $\Delta_{x_0}=R$. (3) With the same assumptions as in (2), let V denote the linear subspace of $\hat{\mathfrak{S}}$ spanned by

$$\{\Psi[\mathfrak{g}_1,\mathfrak{g}_2]-[\Psi(\mathfrak{g}_1),\Psi(\mathfrak{g}_2)]:\mathfrak{g}_1,\mathfrak{g}_2\in\hat{\mathfrak{G}}\}.$$

Then the Lie algebra $\hat{\Sigma}$ of the holonomy group Σ at x_0

is the minimal linear subspace containing $\mathbf{V}$ and invariant under Ad Δ_{x_0}, or what is the same,

$$\hat{\mathbf{V}}+[\Psi(\hat{\mathbf{G}}), \hat{\mathbf{V}}]+[\Psi(\hat{\mathbf{G}}), [\Psi(\hat{\mathbf{G}}), \hat{\mathbf{V}}]]+\cdots.$$

K. Yano (Hong Kong)

6002:

Hlavatý, V. The holonomy group. II. The Lie group induced by a tensor. J. Math. Mech. 8 (1959), 597–622.

This paper is a continuation of J. Math. Mech. 8 (1959), 285–307 [MR **21** #899]. The first part describes the construction of the Lie group induced by a tensor. Let $T^{\Phi_p\Lambda}_{\Psi_q\Omega}$ be a given tensor in an n-dimensional affinely connected space L_n, where the capital Greek indices Φ_p, Ψ_q, Δ, Ω are used instead of $\varphi_1\cdots\varphi_p$, $\psi_1\cdots\psi_q$, $\lambda_1\cdots\lambda_u$, $\omega_1\cdots\omega_u$. The sth covariant derivative of $T^{\Phi_p\Lambda}_{\Psi_q\Omega}$ will be denoted by $T^{\Phi_p\quad\Lambda}_{\Psi_{q+s}\Omega}$. Let $\underset{s}{U}{}^{\Psi_{q+s}}_{\Phi_p}$ be any arbitrary tensor and put $L^{\Lambda}_{\Omega}(\underset{s}{U})=T^{\Phi_p\quad\Lambda}_{\Psi_{q+s}\Omega}\underset{s}{U}{}^{\Psi_{q+s}}_{\Phi_p}$. Next consider the matrix $((T))$ whose columns are $((T_0))$, $((T_1))$, $\cdots$, while $((T_s))$ means the infinite matrix $((\cdots L^{\Lambda}_{\Omega}(\underset{s}{U})\cdots))$ of which each row contains one arbitrary tensor $\underset{s}{U}$. When the rank of $((T))$ is r, then we have a set of r tensors $L_{a\Omega}{}^{\Lambda}$ which define r linearly independent operators $X_a=L_{a\Omega}{}^{\Lambda}y^{\Omega}\partial/\partial y^{\Lambda}$ and it is proved that these define a Lie group $G_r(T)$ (of r parameters) which is called the Lie group induced by the tensor $T^{\Phi_p\Lambda}_{\Psi_q\Omega}$, where y^{Ω} is a tensor $y^{\omega_1\cdots\omega_u}$. The second part deals with the application to the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$, which may be identified with $T^{\Phi_p\Lambda}_{\Psi_q\Omega}$ for $p=0$, $q=2$, $u=1$. The Lie group induced by the curvature tensor is the holonomy group $G_r(R)$. The third part introduces some basic notation and auxiliary results concerning $G_r(R)$ and $G_r(K)$ which the author will use in the next paper of this series. $K_{\omega\mu\lambda}{}^{\nu}$ means the curvature tensor of a Riemannian space.

A. Kawaguchi (Sapporo)

6003:

Kostant, Bertram. On holonomy and homogeneous spaces. Nagoya Math. J. 12 (1957), 31–54.

Let G be a compact connected Lie group and K a closed subgroup. The purpose of the paper is mainly to consider questions of holonomy when G/K is provided with an arbitrary invariant metric, not necessarily a natural one, and to see how reducibility properties change when we change from one metric to another.

In § 2 of the paper the author proves: Let G/K be given any homogeneous Riemannian metric. Let $\mathfrak{g}$ and $\mathfrak{k}$ be respectively Lie algebras of G and K, and let $\mathfrak{p}$ be any complement to $\mathfrak{k}$ such that $\mathfrak{p}$ admits a strictly invariant bilinear form (X, Y). Let $\{X, Y\}$ be the bilinear form on $\mathfrak{p}$ given by the metric tensor. Let $S:\mathfrak{p}\to\mathfrak{p}$ be defined by $(SX, Y)=\{X, Y\}$. Extend S to $\mathfrak{g}$ by defining $S=0$ on $\mathfrak{k}$. Now for any $Z\in\mathfrak{g}$ let D_Z be the operator on $\mathfrak{p}$ defined by $D_ZY=[Z, Y]_{\mathfrak{p}}$ for all $Y\in\mathfrak{p}$. Then the linear holonomy algebra is the Lie algebra generated by all operators on $\mathfrak{p}$ of the form $D_Z+S^{-1}D_ZS-S^{-1}D_{SZ}$, where $Z\in\mathfrak{g}$.

At the beginning of § 3, the author gives an example of a case where $\mathfrak{g}$ or G is simple yet G/K is reducible, which is a counter-example to the conjecture of Nomizu. In the later part of § 3 he proves the following two theorems and determines the linear holonomy group and the linear holonomy algebra for an arbitrary invariant metric. (1) Let C be an arbitrary permissible metric (that is, a positive definite bilinear form on $\mathfrak{p}$ which is invariant under $\mathrm{ad}_{\mathfrak{p}}\, K$). Let $\mathfrak{s}_0(C)$ be the corresponding holonomy algebra, so that $\exp\mathfrak{s}_0(C)=\tau_0(C)$ is the restricted linear holonomy group. Let $\psi_0(C)$ be the full linear holonomy group. Then $\mathrm{ad}_{\mathfrak{p}}\, K\subseteq\psi_0(C)$ and in fact $\psi_0(C)=\mathrm{ad}_{\mathfrak{p}}\,(K)\cdot\tau_0(C)$. (2) A subspace $\mathfrak{p}_1\subseteq\mathfrak{p}$ is invariant under $\tau_0(C)$ if and only if it is invariant under $\psi_0(C)$. Furthermore the elements of $\mathfrak{p}_1$ are fixed under the action of $\tau_0(C)$ if and only if they are fixed under the action of $\psi_0(C)$.

Let B be a strictly invariant bilinear form on $\mathfrak{p}$, that is, a positive definite bilinear form on $\mathfrak{p}$ for which D_X is skew-symmetric for all $X\in\mathfrak{g}$, and let C be any permissible metric. Let $\mathfrak{p}=\sum_{i=0}^{m}\mathfrak{p}_i(C)$ be the direct sum, where $\mathfrak{p}_0(C)$ is the set of all vectors in $\mathfrak{p}$ which are fixed by $\psi_0(C)$, and $\mathfrak{p}_i(C)$, $i=1, 2, \cdots, m(C)$, are irreducibly invariant subspaces under the action of $\psi_0(C)$. In § 4, the author proves: (1) $C<B$ if and only if S leaves the subspaces $\mathfrak{p}_i(B)$ invariant, $i=0, 1, \cdots, m(B)$. (2) $B<C$ whenever S leaves the subspaces $\mathfrak{p}_i(C)$ invariant. Conversely, if $B<C$ we may find a permissible strictly invariant metric B' on $\mathfrak{p}$ which is strongly equivalent to B and which is such that S' leaves the spaces $\mathfrak{p}_i(C)$ invariant. (3) If we assume $\mathfrak{p}\cap\mathfrak{c}=0$, where $\mathfrak{c}$ is the centre of $\mathfrak{g}$, then $B<C$ if and only if S leaves the subspaces $\mathfrak{p}_i(C)$ invariant. (4) Assume $\mathrm{ad}_{\mathfrak{p}}\, K$ has inequivalent representations. Then any two permissible metrics C_1 and C_2 are weakly equivalent. That is, the decomposition $\mathfrak{p}=\sum_{i=1}^{m}\mathfrak{p}_i(C)$ is independent of the metric C.

§ 5 and § 6 are devoted to the applications of these results. To quote one: Assume $\chi(G/K)\neq 0$ and that $\mathfrak{g}$ acts effectively on G/K. Then with respect to any invariant metric, G/K is irreducible if and only if $\mathfrak{g}$ is simple.

K. Yano (Hong Kong)

6004:

Reinhart, Bruce L. Foliated manifolds with bundle-like metrics. Ann. of Math. (2) **69** (1959), 119–132.

A foliated manifold is one on which there exists a p-dimensional $(0<p<n)$ completely integrable field of elements of contact (various examples are given). This structure can be extended to an almost product structure, but in general the complementary field will not be completely integrable. The foliated structure generalizes that which the author has previously studied [Trans. Amer. Math. Soc. **88** (1958), 243–276; MR **21** #3687]. In the present paper he studies the geometry and topology of foliated manifolds, while in the paper reviewed below he returns to the study of the Laplacian.

In addition to the foliation, there is postulated the existence of a "bundle-like" Riemannian metric, which locally is very similar to a product metric. Such a metric always exists for the special case of a fibre space. By a construction using the geodesics which are orthogonal to the leaves (the maximal connected integral manifolds of the foliation), it is shown that any leaf is covered by all nearby leaves (assuming completeness). Assuming "regularity", this implies that the manifold is a fibre space over a complete manifold B—cf. the theorem of de Rham that a complete, simply connected Riemannian manifold with reducible holonomy group is a global product.

M. P. Gaffney (Washington, D.C.)

6005:

Reinhart, Bruce L. Harmonic integrals on foliated manifolds. Amer. J. Math. **81** (1959), 529–536.

Let M be a foliated manifold [see paper reviewed above],

with an orientable foliation given by the q-form Θ. Then it is possible to cover M with neighborhoods whose coordinates $(x^1, \cdots, x^p, y^1, \cdots, y^q)$ have the property that the integral manifolds of Θ are given locally by $y^1=c^1$, $y^2=c^2$, $\cdots$, $y^q=c^q$, and that in addition the partial Jacobians in the x variables of overlapping neighborhoods are $+1$. The foliation leads to a type decomposition of differential forms and of exterior differentiation d. A form representable locally as $\sum \phi_{a_1\cdots a_s}(y)dy^{a_1}\cdots dy^{a_s}$ (coefficient independent of x) is called a base-like form. For such forms $d''^2=0$, leading to the base-like cohomology groups. If M is compact and has a "bundle-like" Riemannian metric, a base-like Laplacian is defined. A Green's operator for Δ'' is constructed by modifying a method used earlier by the author [paper cited above]. A corollary is that the cohomology groups of base-like differential s-forms have finite dimension b^s, and $b^s=b^{q-s}$.

M. P. Gaffney (Washington, D.C.)

6006:

Speranza, Francesco. Sulle trasformazioni che posseggono un gruppo di coppie di corrispondenze in sè. II, III. Boll. Un. Mat. Ital. (3) **14** (1959), 10–27, 42–56. (English summary)

This is a sequel to Part I [same Boll. **13** (1958), 486–496; MR **21** #901]. It considers a number of special cases of groups of "couples of mappings relative to a transformation T". In the first example we have two planes F and $\bar{F}$, in which the following isomorphic groups operate: $x=x'+\lambda$, $y=y'$; $\bar{x}=\bar{x}'+a\lambda$, $\bar{y}=\bar{y}'$. It is shown that these are the groups of a couple of mappings relative to the transformation: $\bar{x}=\psi(y)+ax$, $\bar{y}=\varphi(y)$. Other examples are other groups of motion, groups of similitude, and groups of homography of various types.

C. B. Allendoerfer (Seattle, Wash.)

6007:

Gu, Čao-hao. On the global imbedding of spaces of affine connection into an affine space. Sci. Record (N.S.) **2** (1958), 7–9. (Russian)

Starting from a (local) imbedding theorem by G. Laptev [Dokl. Akad. Nauk SSSR **41** (1943), 315–317; MR **6**, 117] three theorems on imbedding are stated. The second is: every n-dimensional differentiable manifold R^n of class s $(s\geqq 3)$ with a symmetric affine connection with coefficients $\Gamma_{\beta\gamma}{}^{\alpha}$ of class C^r $(1\leqq r\leqq s-2)$ can be regularly and properly C^s-imbedded in an affine space E of $n(n+5)/2$ dimensions such that the second order osculating space at every point of the imbedding has $n(n+3)/2$ dimensions. The third theorem then states that every such R^n can be C^s-imbedded in an E of $n(n+5)/2$ dimensions, with a reference to related work by J. Nash on Riemannian manifolds imbedded in Euclidean space [Ann. of Math. (2) **63** (1956), 20–63; MR **17**, 782].

D. J. Struik (Cambridge, Mass.)

6008:

Matschinski, Matthias. De la géométrie sur la surface d'un polyèdre et dans l'hypersurface d'un polytope. Acad. Roy. Belg. Bull. Cl. Sci. (5) **45** (1959), 78–101.

The author examines certain properties of polyhedral surfaces and hypersurfaces commonly thought of as intrinsic, and shows that in fact they depend on the underlying space. The Levi-Civita method of parallel displacement depends on the continuity of the first derivatives, and hence cannot be applied universally in the case of a polytope. The author's goal is a statement of conditions governing 'total invariance' of his formula for parallel displacement.

F. A. Sherk (Toronto, Ont.)

6009:

Rembs, Eduard. Über die Verbiegbarkeit der Rinnen. Math. Z. **71** (1959), 89–93.

A "groove" [=Rinne] is a three times differentiable surface R containing a plane curve K such that the plane through K supports R along K and that K divides R into two regions of opposite non-vanishing Gauss curvature. Th. 1: Two analytic isometric grooves are congruent. Th. 3: A groove is non-deformable. Th. 5: Let K be closed. Then every surface isometric to R which has a plane convex closed parabolic curve is congruent to R. {The reviewer does not understand Th. 5 and the application of Th. 1 in the proof of Th. 3.}

P. Scherk (Toronto, Ont.)

6010:

Grotemeyer, K. P. Zur Flächentheorie im Grossen. III. Flächenstücke mit Kanten. Arch. Math. **10** (1959), 216–220.

[For parts I and II, see same Arch. **9** (1958), 117–122, 382–388; MR **21** #2280, #2281.] An "edge" [=Kante] is a pair of strips through the same curve. Consider finitely connected surfaces $\mathfrak{F}$ and $\mathfrak{F}^*$ with a finite number of edges but satisfying suitable smoothness conditions otherwise. The author rewrites the symmetrized Herglotz integral formula for isometric $\mathfrak{F}$ and $\mathfrak{F}^*$. He extends Blaschke's integral formula to infinitesimal deformations of $\mathfrak{F}$ [cf. Efimow, *Flächenverbiegung im Grossen*, Akademie-Verlag, Berlin, 1957; MR **19**, 59; pp. 156 and 74]. He also shows that his own integral formula for Minkowski's problem remains valid if edges are admitted [see part I]. These results make it possible to extend uniqueness and rigidity theorems proved by means of these formulas to surfaces with edges.

P. Scherk (Toronto, Ont.)

6011:

Shing, Ding-kia. On the symmetric properties in some Finsler spaces. Acta Math. Sinica **9** (1959), 191–198. (Chinese. English summary)

Let O be a point in a Finsler space F_n. Given a point M sufficiently close to O, there exists a unique point M' such that the geodesic arc $\widehat{MM'}$ joining M and M' contains O as its mid-point. The transformation $M\to M'$ induces a transformation σ of the space of the tangent vectors at various points. This σ carries a tangent vector $\mathbf{x}$ at M to a tangent vector at M'. The correspondence $\mathbf{x}\to-\sigma(\mathbf{x})$ is called the symmetric displacement about O. On the other hand, the parallel displacement along the geodesic $\widehat{MM'}$ carries $\mathbf{x}$ to a tangent vector $\mathbf{y}$ at M'. As M approaches the point O, the difference Δ between $\mathbf{y}$ and $-\sigma(\mathbf{x})$ approaches zero. When F_n is Riemannian, E. Cartan [*Leçons sur la géométrie des espaces de Riemann*, 2d ed. Gauthier-Villars, Paris, 1951; MR **13**, 491] showed that Δ is always an infinitesimal of order 3 (taking the distance between O and M as the principal infinitesimal), and that Δ is an infinitesimal of order higher than 3 if and only if the space is

symmetric. In this paper, the author generalizes these results to Finsler spaces and proves the following: (i) Δ is always an infinitesimal of order 3 if and only if the torsion tensor has zero covariant derivatives; (ii) Δ is an infinitesimal of at least order 4 if and only if both the torsion and curvature tensors have zero covariant derivatives. Moreover, in this case Δ actually vanishes.

H. C. Wang (Evanston, Ill.)

6012:

Moór, Arthur. Über nicht-holonome allgemeine metrische Linienelementräume. Acta Math. **101** (1959), 201–233.

In a recent paper [Acta Sci. Math. Szeged **19** (1956), 85–120; MR **19**, 980] the author introduced a theory of n-dimensional manifolds whose given metric tensor $g_{ij}(x^k, x'^k)$ $(i, j, k=1, \cdots, n;\ x^k, x'^k$ positional and directional coordinates, respectively) need not necessarily represent the second directional derivatives of a scalar function $\frac{1}{2}F^2(x^k, x'^k)$ (thus generalizing Finsler geometry). In the present paper he studies the corresponding non-holonomic geometry which results from the imposition of m constraints of the form $G_\mu(x^k, x'^k)=0$ $(\mu=1, \cdots, m)$, and a number of theorems are obtained which are natural generalizations of the work of J. L. Synge [Math. Ann. **99** (1928), 738–751] and the reviewer [Math. Nachr. **11** (1954), 61–80; MR **15**, 898]. Besides the autoparallels of the (unconstrained) geometry and the extremals resulting from the fundamental function defined by $\frac{1}{2}f^2(x^k, x'^k)=g_{ij}x'^ix'^j$, a further class of curves, the so-called trajectories ("Bahnkurve") have to be considered. The latter satisfy the equations of constraint while their principal normals are transversal to certain $(n-m)$-dimensional local non-holonomic submanifolds defined by the constraints. These conditions determine the trajectories uniquely and give rise to a non-holonomic connection such that they are autoparallel with respect to this connection. A theory of curvature (including the Bianchi identities, "autoparallel deviation" of neighboring trajectories—being the analogue of the "geodesic deviation" of Levi-Civita) based on this connection is developed in detail. Necessary and sufficient conditions for the coincidence of the trajectories with the extremals of the problem of Lagrange corresponding to the integrand $f(x^k, x'^k)$ and the constraints are derived. In part II the author considers the more general case in which the directional arguments appearing in the equations of constraint do not necessarily coincide with the tangent vectors of the corresponding trajectories.

H. Rund (Durban)

6013:

★Busemann, Herbert. Axioms for geodesics and their implications. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 146–159. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi+488 pp. $12.00.

This is an expository account of the theory of G-spaces established by the author [H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955; MR **17**, 779]. Roughly speaking, a G-space is a finitely compact, convex metric space in which two sufficiently close points determine one and only one geodesic. (By a geodesic in a metric space, we mean a locally isometric image of the entire real line.) By using purely continuous arguments, the following are discussed: G-spaces with negative curvature, G-surfaces with given geodesics, theory of parallels, and characterizations of the classical geometries. The author also mentions the theory of metric surfaces of A. D. Aleksandrov [*Vnutrennyaya geometriya vypuklyh poverhnosteĭ*, OGIZ, Moscow-Leningrad, 1948; translated as *Die innere Geometrie der konvexen Flächen*, Akademie-Verlag, Berlin, 1955; MR **10**, 619; **17**, 74].

H. C. Wang (Evanston, Ill.)

PROBABILITY

See also 5824, 5825, 5844, 6042, 6071, 6136, 6144, 6296, 6304.

6014:

★Burnside, William. Theory of probability. Dover Publications, Inc., New York, 1959. xxx+106 pp. Paperbound: $1.00.

Unabridged and unaltered republication of the 1928 edition [Cambridge Univ. Press].

6015:

Varadarajan, V. S. A useful convergence theorem. Sankhyā **20** (1958), 221–222.

Eine Folge (P_n) von Wahrscheinlichkeitsverteilungen im k-dimensionalen Raum X ist dann und nur dann schwach konvergent, wenn für jede lineare reellwertige auf X definierte Funktion λ die Folge der eindimensionalen Wahrscheinlichkeitsverteilungen $P_n\lambda^{-1}$ schwach konvergiert.

K. Krickeberg (Aarhus)

6016:

Lévy, Paul. Esquisse d'une théorie de la multiplication des variables aléatoires. Ann. Sci. École Norm. Sup. (3) **76** (1959), 59–82.

The following problems are posed (U and V independent). (A) Given X and U, find all V such that $X=UV$. (B) Given X, find all U and V such that $X=UV$. (C) Find conditions on U and V so that UV is symmetric. (D) Find all infinitely divisible laws in the sense of multiplication. (E) If $X=UV$ and the moment problem is determinate for X, is it also determinate for U? (F) Is the determinateness of the moment problem for X equivalent to that of $|X|$, or that of X^2? (G) If the moments of X'^2 majorize those of V^2, does the determinateness of the moment problem for X'^2 imply that for V^2? Some lines of attack are indicated. The following problem of Wintner [Publ. Inst. Statist. Univ. Paris **6** (1957), 327–336; MR **20** #6151] is discussed: if U is a unit normal random variable and V is a non-negative one independent of U, when is UV infinitely divisible (in the classical sense)? A gap in Wintner's proof is pointed out. Is there an infinitely divisible symmetric law which is a normal stratification but some "rth self-convolution" of which is not? The author transmits the following corrections: p. 61, formula (4), read

$|u|^{-1}du$, $|u'|^{-1}du'$; p. 79, the condition on $\mu^*(x)$ in (14) should be "$\int_0^\infty (1+x^2)^{-1}d\mu^*(x) < \infty$" instead of "bounded in $[0, \infty)$".

K. L. Chung (Syracuse, N.Y.)

6017:

Kemeny, John G. A probability limit theorem requiring no moments. Proc. Amer. Math. Soc. **10** (1959), 607–612.

Let the random variable X take integral multiples a_i of a rational number with probabilities $p_i > 0$. Let the (positive) interval of convergence of $f(s) = \sum_i p_i s^{a_i}$ be (s_1, s_2), where $0 \leqq s_1 \leqq 1 \leqq s_2 \leqq \infty$; $g(s) = sf'(s)/f(s)$. Then g is increasing in (s_1, s_2) and approaches limits t_1 and t_2 at the ends. Define h as the inverse function to g in (t_1, t_2) and extend it horizontally at either end. Now if S_n is the sum of n independent random variables with the distribution of X, b an integer and a, a', t such that the quotient on the left side below is finite and positive for all sufficiently large values of n, then

$$\lim_{n\to\infty} P\{S_{n-b} = nt+a\}/P\{S_n = nt+a'\} = h(t)^{a'-a}/f[h(t)]^b.$$

This is proved by an interesting reduction to the special case of the reviewer and Erdös [Mem. Amer. Math. Soc. no. 6 (1951); MR **12**, 722], where $t=0$ and the limit is 1 when either $E(X)=0$ or $E(|X|+X)=E(|X|-X)=\infty$.

K. L. Chung (Syracuse, N.Y.)

6018:

Zaremba, S. K. On necessary conditions for the central limit theorem. Math. Z. **70** (1958), 281–282.

This note gives an interesting and elementary proof, based on direct probabilistic arguments of the following result of Feller concerning the necessary conditions for the central limit theorem: Let $\{x_k\}$ be a sequence of independent random variables, and $\{a_n\}$ and $\{b_n\}$ numerical sequences with $a_n > 0$ $(n = 1, 2, \cdots)$. If the distribution of $a_n^{-1}(x_1 + \cdots + x_n) - b_n$ converges to a normal distribution with zero mean and unit variance, and if the random variables $\{a_n^{-1}x_k\}$ $(k = 1, \cdots, n)$ are infinitesimal, then, for every positive number η, $P(|x_1| > \eta a_n) + \cdots + P(|x_n| > \eta a_n) \to 0$ as $n \to \infty$.

G. Kallianpur (Bloomington, Ind.)

6019:

Erdős, Paul; and Rényi, Alfréd. On the central limit theorem for samples from a finite population. Magyar Tud. Akad. Mat. Kutató Int. Kőzl. **4** (1959), 49–61. (Hungarian and Russian summaries)

Let $\{a_{nk}, 1 \leqq k \leqq n\}$ be real numbers with $\sum_{k=1}^n a_{nk} = 0$,

$$D_{ns} = \sqrt{\left(\frac{s}{n}\left(1-\frac{s}{n}\right)\sum_{k=1}^n a_{nk}^2\right)} \quad (1 \leqq s \leqq n).$$

Let $N_{ns}(x)$ be the number of sums of s distinct terms of $\{a_{nk}, 1 \leqq k \leqq n\}$ that are $\leqq xD_{ns}$. If $s_n \leqq n/2$ and the Lindeberg condition

$$\lim_{n\to\infty} \frac{1}{D_{nn}^2} \sum_{|a_{nk}| > \varepsilon D_{ns_n}} a_{nk}^2 = 0$$

holds, then

$$\lim_{n\to\infty} N_{ns_n}(x) \Big/ \binom{n}{s_n} = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^x e^{-y^2/2} dy.$$

This is a form of the combinatorial central limit theorem (sampling without replacement) which includes a previous result of Madow.

K. L. Chung (Syracuse, N.Y.)

6020:

★Gnedenko, B. W.; und Kolmogorov, A. N. Grenzverteilungen von Summen unabhängiger Zufallsgrössen. Wissenschaftliche Bearbeitung der deutschen Ausgabe: Prof. Dr. Josef Heinhold. Mathematische Lehrbücher und Monographien, II. Abt., Bd. IX. Akademie-Verlag, Berlin, 1959. viii + 279 pp. DM 44; $10.48.

The original Russian edition of this work [*Predel'nyh raspredeleniya dlya summ nezavisimyh slučaĭnyh veličin*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949] was reviewed in MR **12**, 839; a Hungarian translation [Akadémiai Kiadó, Budapest, 1951] is listed in MR **14**, 294 and an English translation [Addison-Wesley, Cambridge, Mass., 1954] in MR **16**, 152. The present translation is based on the Russian edition as corrected by the authors.

6021:

Zlatev, Iv. Über die asymptotische Verteilung sich in einem Rechteck bewegender Punkte. C. R. Acad. Bulgare Sci. **12** (1959), 109–112. (Russian summary)

Consider a particle which moves throughout a 2-dimensional rectangle, $0 \leqq x \leqq a$, $0 \leqq y \leqq b$. Let (x_t, y_t) denote the particle's position at time $t \geqq 0$. Assume that, for all $t \geqq 0$, the velocity components are equal to $(\dot{x}_0, \dot{y}_0)$, say. At the sides of the rectangle, the point reflects, keeping the angles of incidence and reflection equal. The author assumes that $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ is a random vector with an absolutely continuous distribution function over $[0, a] \times [0, b] \times R_1 \times R_1$ where R_1 denotes the real line. The author proves, using simple methods, that the distribution of (x_t, y_t) converges to the uniform distribution over the rectangle, as $t \to \infty$. {There are a few misprints in the paper. In particular, the limits of integration in the middle and at the bottom of p. 111 are incorrect. For example, $\xi_{k,2}^{(t)}/t$ should read $(\xi_{k,2}^{(t)} - x_0)/t$.}

R. Pyke (New York, N.Y.)

6022:

Getoor, R. K. Markov operators and their associated semi-groups. Pacific J. Math. **9** (1959), 449–472.

This paper continues an investigation on functional of a temporarily homogeneous Markov process initiated in an earlier paper [see R. K. Getoor, same J. **7** (1957), 1577–1591; MR **20** #1359]. The results of this earlier paper are shown to hold without a condition (P_3) previously assumed. Otherwise the assumptions are as before. Let G be an open subset of the range space X of the process $\{x(t)\}$ and V a non-negative measurable function on X. The basic interest is in the expectation

$$K(V, G; t, x, A) = E\left[\exp\left(-\int_0^t V(x(\tau))d\tau\right); x(\tau) \in G,\right.$$
$$\left. 0 \leqq \tau \leqq t, x(t) \in A \,|\, x(0) = x\right]$$

so that one only considers paths that never leave G. This expectation is shown to generate a semigroup in that

$$\int K(V, G; t, x, dy)K(V, G; s, y, A) = K(V, G; t+s, x, A).$$

Behaviour of this semigroup at the boundary of G is examined. There is a discussion of the special case in which the infinitesimal generator of the semigroup directly

associated with $x(t)$ is a local operator. Spectral properties of the semigroups introduced are studied.

M. Rosenblatt (Providence, R.I.)

6023:

Blumenthal, R. M.; and Getoor, R. K. The asymptotic distribution of the eigenvalues for a class of Markov operators. Pacific J. Math. **9** (1959), 399–408.

This paper is an extension of the one by Getoor reviewed above. Let $p(t, x, A) = \int f(t, x, y)m(dy)$ be the transition function of the Markov process $\{x(t)\}$. Let G be an open subset of X with $m(\partial G) = 0$ where ∂G is the boundary of G. If for each $t > 0$, $f(t, x, y) \in L_2(G \times G, m \times m)$, and $f(t, x, y) = f(t, y, x) > 0$ for all t, x, y, Getoor in the paper referred to has shown that the spectrum of the semigroup generated by $K(V, G; t, x, A)$ is discrete. The asymptotic distribution of the eigenvalues of this semigroup is obtained in this paper for a wide class of processes.

M. Rosenblatt (Providence, R.I.)

6024:

Masani, Pesi. Sur la fonction génératrice d'un processus stochastique vectoriel. C. R. Acad. Sci. Paris **249** (1959), 360–362.

Cette note fait suite aux articles de Wiener et Masani [Acta Math. **98** (1957), 111–150; **99** (1958), 93–137; MR **20** #4323, #4325]. La fonction génératrice d'un processus stochastique vectoriel, faiblement stationnaire et non déterministe, est de la forme $\Phi(e^{i\theta}) \sim \sum_0^\infty C_k e^{ik\theta} \in L_2{}^{0+}$; l'auteur montre qu'on peut l'écrire $G^{1/2}\Omega(e^{i\theta})$, où G est la matrice d'erreur, $\Omega(e^{i\theta}) \in L_2{}^{0+}$, et $\Omega(e^{i\theta})$ inversible p.p.

J. P. Kahane (Montpellier)

6025:

Kampé de Fériet, Joseph; and Frenkiel, François. Estimation de la corrélation d'une fonction aléatoire non stationnaire. C. R. Acad. Sci. Paris **249** (1959), 348–351.

Random functions $f(t, w)$ with t real, $w \in \Omega$, are considered that satisfy appropriate measurability and moment conditions. Let $R_T(h)$ be the following mean of covariances:

$$R_T(h) = T^{-1} \int_{|h|/2}^{T-|h|/2} \operatorname{cov}\,(f(t-\tfrac{1}{2}h, w), f(t+\tfrac{1}{2}h, w))dt.$$

The authors' interest is in processes for which the limit $R(h) = \lim_{t\to\infty} R_T(h)$ exists. Several nonstationary processes with this property are exhibited.

M. Rosenblatt (Providence, R.I.)

6026:

Lloyd, S. P. A sampling theorem for stationary (wide sense) stochastic processes. Trans. Amer. Math. Soc. **92** (1959), 1–12.

Let $x(t)$, $-\infty < t < \infty$, be a wide sense stationary stochastic process. A necessary and sufficient condition under which the random variables $x(t)$ are determined linearly by the random variables $x(nh)$, $-\infty < n < \infty$, $h > 0$ fixed, are obtained. The condition is that there be a support Λ of the spectral distribution of the process $x(t)$ whose translates $\{\Lambda - nh^{-1}, -\infty < n < \infty\}$ are mutually disjoint. Further information on the form of the linear representation of $x(t)$ in terms of the $x(nh)$ is obtained.

M. Rosenblatt (Providence, R.I.)

6027:

Ray, Daniel. Resolvents, transition functions, and strongly Markovian processes. Ann. of Math. (2) **70** (1959), 43–72.

Let X be a compact Hausdorff space, C the space of real continuous functions on X, and B the smallest σ-field of subsets of X for which the functions of C are measurable.

Let R_λ be a resolvent on C in the sense that $0 \leq \lambda R_\lambda f \leq 1$ whenever $0 \leq f \leq 1$ with $\lambda R_\lambda f \equiv 1$ if $f \equiv 1$, and $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$. Suppose $C_1 \subset C$ is a subset of positive functions, separating points in X, such that $\lambda R_{\lambda+1} f \leq f$. Then there is a unique transition function $P_t(x, E)$ on X such that $P_t f(x) \equiv \int f(y) P_t(x, dy)$ is right continuous in t and

$$\int_0^\infty e^{-\lambda t} P_t f(x) dt = R_\lambda f(x) \quad (\lambda > 0).$$

The range of R_λ is not supposed to be dense in X. As a consequence $P_t f(x)$ need not approach $f(x)$ as $t \to 0$, for some $f \in C$. A point x is called a branching point if there is a measure μ on B, not concentrated at x, with $\mu(X) = 1$, and $f(x) \geq \int f(y)\mu(dy)$, $f \in C_1$. Let X_b denote the set of branching points. Then $P_t f(x) \to f(x)$ as $t \to 0$ for each $f \in C$ if and only if $x \in X - X_b$. For $x \in X_b$ there is a measure μ_x, not concentrated at x, such that $\lim P_t f(x) = \int f(y) \mu_x(dt)$, $f \in C$. Further, $P_t(x, X - X_b) = 1$ for all x and $t > 0$.

The further properties of the transition function are given in terms of probability theory. Let X be a compact metric space and let C_1 be countable. If there is a Markov process on X with transition function $P_t(x, E)$, defined above, then there is a Markov process (Ω, F, P, Y_t) such that (1) $Y_t(\omega) \in X - X_b$, (2) $Y_t(\omega) = \lim Y_s(\omega)$ if s is rational and $s \to t$, (3) with probability one $Y_t(\omega)$ is right continuous and has also limits from the left, and (4) the process is strongly Markovian. Slightly weaker results hold in the general case. Further, generalizations are considered in which X is completed relative to a uniform structure determined by positive bounded measurable functions on X. If X_R is the subset of the completed space $\bar{X}$ at which $R_\lambda f$ is continuous for f in a certain space $\bar{M}$, and if X_R is a measurable subset of $\bar{X}$, then for the transition function defined above, $P_t(x, X_R) = 1$, $x \in X_R$, $t > 0$. If (Ω, F, P, Y_t) is defined as above and if the initial distribution of the process is concentrated on X_R, then with probability one, Y_t is in X_R for all $t \geq 0$.

E. Hille (New Haven, Conn.)

6028:

Dynkin, E. B. The natural topology and excessive functions connected with a Markov process. Dokl. Akad. Nauk SSSR **127** (1959), 17–19. (Russian)

Let $P(t, x, \Gamma)$ be a stationary transition function on a measurable space. A measurable function f is called excessive if it is non-negative and if $\int P(t, x, d\gamma) f(\gamma) \leq f(x)$, with equality in the limit, when $t \to 0$. Interrelations between this definition and a more probabilistic one are detailed. If the measurable space is a locally compact Hausdorff space, with measurable sets the Borel sets, then under suitable restrictions on the transition functions (or on the corresponding Markov processes) the excessive functions must be lower semi-continuous, as in the classical case when the excessive functions are the non-negative superharmonic functions. Define a new topology on the space by defining a Borel set to be a neighborhood of a point if almost every probability path from the point

lies in the set for a non-degenerate time interval. Then the new topology is the least-fine topology in which all excessive functions, both for the given process and its subprocesses, are continuous. Proofs are omitted. In the latter formulation the new topology was introduced by H. Cartan [Ann. Univ. Grenoble Sect. Sci. Math. Phys. (N.S.) **22** (1946), 221–280; MR **8**, 581]. The probabilistic significance in classical cases was given by the reviewer [Trans. Amer. Math. Soc. **77** (1954), 86–121; **80** (1955), 216–280; MR **16**, 269; **18**, 76]. Excessive functions were first defined by Hunt [Illinois J. Math. **1** (1957), 44–93; MR **19**, 951]. *J. L. Doob* (Urbana, Ill.)

6029:

Gilbert, Edgar J. On the identifiability problem for functions of finite Markov chains. Ann. Math. Statist. **30** (1959), 688–697.

Let $\{X_n\}$ be a stationary Markov chain with a stationary absolute distribution and a finite set of states $1, 2, \cdots, N$. Let f be a function on the integers $1, 2, \cdots, N$ whose arguments are integers $0, 1, \cdots, D-1$, where $D \leq N$. Then the process $\{Y_n\}$, where $Y_n = f(X_n)$, is called a function of a finite Markov chain. If s and t are finite sequences of states of $\{Y_n\}$, of lengths L_1 and L_2, and ε is a state of $\{Y_n\}$, then $p(s\varepsilon t)$ is the probability that any L_1+L_2+1 successive values of $\{Y_n\}$ are the states in s followed by ε followed by the states in t. For each state ε of $\{Y_n\}$ let $n(\varepsilon)$ be the largest integer n such that there are $2n$ finite sequences $s_1, \cdots, s_n$ and $t_1, \cdots, t_n$ such that the $n \times n$ matrix $(p(s_i\varepsilon t_j))$ is nonsingular. Lemma 1. If $\{Y_n\}$ is a function of a finite Markov chain, then $\sum n(\varepsilon) \leq N$. We say that $\{Y_n\}$ is a regular function of a finite Markov chain if $\sum n(\varepsilon) = N$. Theorem 1. If $\{Y_n\}$ has D states and is a regular function of a Markov chain having N states, then the entire distribution of $\{Y_n\}$ is determined by the set of functions

$$\{p(s) : \text{length of } s \leq 2(N-D+1)\}.$$

If the process is not regular a modified version of the theorem holds. Blackwell and Koopmans [Ann. Math. Statist. **28** (1957), 1011–1015; MR **20** #5525] had previously obtained the upper bound $2N^2+1$. A parametric representation is obtained for all $N \times N$ Markov matrices which give the same distribution for a given regular function of a Markov chain.

T. E. Harris (Santa Monica, Calif.)

6030:

★**Darling, D. A. Étude des fonctionnelles additives des processus markoviens. Le calcul des probabilités et ses applications.** Paris, 15–20 Juillet 1958, pp. 69–80. Colloques Internationaux du Centre National de la Recherche Scientifique, LXXXVII. Centre National de la Recherche Scientifique, Paris, 1959. 196 pp.

Es sei (X, S) ein meßbarer Raum und $x(t)$, $t \geq 0$, ein Markoffscher Prozeß mit Werten in X und stationären Übergangswahrscheinlichkeiten. V sei eine S-meßbare Abbildung von R in den euklidischen R_1. Es existiere für alle $t \geq 0$ $U(t) = \int_0^t V(x(\tau))d\tau$. Es interessiert die Verteilung von $U(t)$. Verfasser gibt für das Problem eine überblicksmäßige, abgerundete Darstellung der bisher erzielten Ergebnisse, welche auf Darling, Doob, Dynkin, Feller, Hasminskiĭ, Kac, Siegert u.a. zurückgehen. Für den Fall, daß X der euklidische R_1 ist, und $x(t)$ ein homogener Prozeß mit unabhängigen Zuwächsen, werden spezielle neue Ergebnisse angegeben. *L. Schmetterer* (Hamburg)

6031:

Maruyama, Gisirô. On the strong Markov property. Mem. Fac. Sci. Kyushu Univ. Ser. A **13** (1959), 17–29.

Some known facts about stopping times for Markov processes and the preservation of the Markov property under optional sampling are given with slight improvements. It is impossible to cite all the pertinent literature in a short space, but a fair sample is D. G. Austin [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 575–578; MR **20** #7350], R. Blumenthal [Trans. Amer. Math. Soc. **85** (1957), 52–72; MR **19**, 468], E. B. Dynkin and A. Juškevič [Teor. Veroyatnost. i Primenen. **1** (1956), 149–155; MR **19**, 469; ibid. **2** (1957), 187–213], and V. Volkonskiĭ [ibid. **3** (1958), 332–350; MR **20** #7344]. The author seems to be unaware of Blumenthal's and Volkonskiĭ's papers.

H. P. McKean, Jr. (Cambridge, Mass.)

6032:

Neveu, Jacques. Processus sous-markoviens stationnaires. C. R. Acad. Sci. Paris **249** (1959), 1447–1449.

Consider (dishonest) Markov chains $C = (p, Q)$, where Q is a countable state space and $p = p_t(i, j)$ $(t > 0, (i, j) \in Q \times Q)$ is a (dishonest) Markov transition function such that $p_{0+}(i, i) = 1$ $(i \in Q)$.

Given 2 such chains C^1 and C^2, $\Delta^{12}_{s,t}(i, j)$ $(s, t > 0, (i, j) \in Q^1 \times Q^2)$ is said to be a form for C^1 and C^2 if it is non-negative and if $\Delta^{12}_{s+s',t+t'} = p^1_s \Delta^{12}_{s',t'} p^2_t$ $(s, s', t, t' > 0)$, and now the statement is that if $Q^2 \supset Q^1$ and if $p^2 \geq p^1$ on $Q^1 \times Q^1$, then there is a form Δ^{12} for C^1 and C^2 and a form Δ^{21} for C^2 and C^1 such that, with the extension $p^1 \equiv 0$ outside $Q^1 \times Q^1$,

$$\text{(1)} \qquad \begin{aligned} p^2_t &= p^1_t + \int_0^t \Delta^{12}_{s,t-s}\, ds \\ \lim_{s \downarrow 0} p^1_s s^{-1}(p^2_s - p^1_s) p^2_t &= \Delta^{12}_{s,t}, \end{aligned} \qquad (i, j) \in Q^1 \times Q^2$$

$$\text{(2)} \qquad \begin{aligned} p^1_t &= p^2_t + \int_0^t \Delta^{12}_{s,t-s}\, ds \\ \lim_{s \downarrow 0} p^2_s s^{-1}(p^2_s - p^1_s) p^1_t &= \Delta^{21}_{s,t}, \end{aligned} \qquad (i, j) \in Q^2 \times Q^1$$

$$\text{(3)} \qquad \lim_{s \downarrow 0} \Delta^{12}_{s,s} p^1_{t-s} = \lim_{s \downarrow 0} p^1_{s-s} \Delta^{21}_{s,t} \qquad (i, j) \in Q^1 \times Q^1.$$

Consider now a chain C^2 such that the class $Q^1 = (i : \lim_{s \downarrow 0} s^{-1}[1 - p^2_s(i, i)] = q_i < \infty)$ of stable states is not void and define a chain $C^1 = (p^1, Q^1)$ with $p^1_t(i, i) = e^{-q_i t}$ $(t > 0,\ i \in Q^1)$ and $p^1_t(i, j) = 0$ otherwise. $Q^2 \supset Q^1$ and $p^2 \geq p^1$ on $Q^1 \times Q^1$, and the existence of forms Δ^{12} and Δ^{21} as in (1), (2), (3) implies that

$$\text{(4)} \qquad \frac{\partial}{\partial t} p^2_t(i, j) \qquad (i, j) \in (Q^1 \times Q^2) \cup (Q^2 \times Q^1)$$

exists and is continuous $(t \geq 0)$, a fact due to D. G. Austin [Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 224–226; MR **16**, 1130]. *H. P. McKean, Jr.* (Cambridge, Mass.)

6033:

Urbanik, K. Filtering of stationary generalized stochastic processes. Sci. Record (N.S.) **2** (1958), 43–45.

A generalized stochastic process (g.s.p.; see Urbanik, Teor. Veroyatnost. i Primenen. **1** (1956), 146–149 [MR **19**, 326] for definition) Φ is called stationary if (1) its "expectation" $E\Phi(\omega, t)$ is constant in t, and (2) its correlation distribution, at (s, t), depends only on $s-t$ (and hence is of the form $B_\Phi^{(s-t)}$, where B_Φ is a distribution). It is known that $B_\Phi^{(t)}$ can then be written as $\int_{-\infty}^{\infty} e^{iut}dH_\Phi(u)$, where H_Φ is a nondecreasing function of polynomial growth. Φ is said to have orthogonal values if B_Φ has its support at 0, and is called a white noise if $B_\Phi=\delta$. Theorem 1: A stationary g.s.p. $\Phi(\omega, t)$ can be obtained as

$$\int_{-\infty}^{\infty} a(t-\lambda)\Psi(\omega, \lambda)d\lambda,$$

where a is a distribution and $\Psi(\omega, t)$ a white noise, $\Leftrightarrow H_\Phi$ is absolutely continuous (the integral is defined in the author's basic paper on g.s.p., Studia Math. **16** (1958), 268–334 [MR **20** #4309]). $a(t)$ is then $(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{iut}\varphi(u)du$, where φ is the Fourier transform of the derivative of H_Φ. Theorem 2: A stationary g.s.p. has orthogonal values $\Leftrightarrow$ it is a finite linear combination of derivatives of a white noise. The distributions a which can arise in theorem 1, i.e., as Fourier transforms of locally square-integrable functions of polynomial growth, are characterized in theorem 3 as distributions for which $a*\bar{a}$ exists, in an appropriate sense. No proofs are given.

J. Feldman (Berkeley, Calif.)

6034:

Cheng, S. L.; and Urbanik, K. On the values at the fixed moment of strictly stationary generalized stochastic processes. Sci. Record (N.S.) **2** (1958), 47–51.

A notion of "taking a value at a point" is defined for a g.s.p. [see review above]; the "value" is then a random variable. The definition is a stochastic version of Łojasiewicz's notion of "value of a distribution at a point". Theorem: Let $\phi(\omega, t)$ be a strictly stationary g.s.p. having a value at t_0; then it has a value at every t. Remark (without proof): The "values", call them $\chi(\omega, t)$, can, by choosing the proper $\chi(\cdot, t)$ representative for each t, be made into a strictly stationary measurable ordinary stochastic process.

J. Feldman (Berkeley, Calif.)

6035:

Urbanik, K. Local characteristics of generalized stochastic processes. Studia Math. **17** (1958), 199–266.

Having previously discussed a notion of "taking a value at t_0" for a g.s.p. [see review above], the author obtains here a classification of local properties which applies to a wider class (but by no means all) of g.s.p. It again reduces, for deterministic processes, to "value of a distribution at a point". Gelfand has proposed the problem of constructing some sort of object ("generalized probability distribution") which would be determined by a g.s.p. near t_0, and would describe the local properties of the g.s.p. sufficiently for recapturing, in the case of a derivative of a process with independent increments, all probabilistic information about the g.s.p. from knowledge of these generalized probability distributions at all t_0. Urbanik's "local characteristics" give a partial solution to this. Reformulating his definition slightly, for simplicity: consider sequences $\langle \alpha_n, \Lambda_n\rangle$ $(n=0, 1, 2, \cdots)$, where α_n is a function from the positive reals to the positive reals, and Λ_n is a measure of total mass 1 on the Stone-Čech compactification of the reals. $\langle \alpha_n, \Lambda_n\rangle$ is identified with $\langle \tilde{\alpha}_n, \tilde{\Lambda}_n\rangle$ if $\Lambda_n=\tilde{\Lambda}_n$ and $\lim_{x\downarrow 0} \tilde{\alpha}_n(x)/\alpha_n(x)=1$. Let $\mathscr{M}$ be the set of measurable stochastic processes $f(\omega, t)$ for which $E|f(\omega, t)|^n$ is locally integrable in t, for all $n\geqq 0$, and $\lim_{h\to 0} E|\Delta_h f(\omega, t)|^n=0$. For any derivative $\Phi(\omega, t)$ of a continuous process in $\mathscr{M}$, the notion of $\langle \alpha_n, \Lambda_n\rangle$ being a local characteristic of Φ at t_0 is defined. In the case of a first derivative, $df(\omega, t)/dt$, the notion reduces to

$$\lim_{h\downarrow 0} \alpha_n(h)\int_{-\infty}^{\infty} \varphi(x)(1+|x|^n)d\,\Pr\{\Delta_h f(\omega, t_0) < x\}= \Lambda_n(\varphi),$$

for $n=0, 1, 2, \cdots$, and each continuous function φ with limits at $+\infty$ and $-\infty$ (Note: The author actually uses $L_n=(1+|x|^n)^{-1}\Lambda_n$ rather than Λ_n in his definition.) The local characteristic at t_0, if it exists, is determined modulo the aforementioned equivalence. Local characteristics are calculated (a) for derivatives of homogeneous Brownian motion processes, and (b) for continuous processes in $\mathscr{M}$, where they amount to the usual probability distributions at t_0. However, a continuous process not in $\mathscr{M}$ may have local characteristics which do not come just from its value at t_0.

Let $\mathscr{X}$ be the class of measurable homogeneous stochastic processes with independent increments and locally integrable sample functions. Let $\mathscr{X}_s$ $(s=1, 2, \cdots)$ be the class of kth derivatives of processes in $\mathscr{X}$; these are then stationary g.s.p. with independent values. By stationarity, local characteristics, if they exist at t_0, exist everywhere, and are the same. Certain small classes of local characteristics are explicitly constructed: (1) singulary, (2) Poissonian, (3) quasipoissonian, (4) Cauchy, (5) uniform. Th. IV.1: Let $\Phi(\omega, t)\in\mathscr{X}_s$, and have local characteristic. Then its characteristic must fall into classes (1)–(5). If $s\geqq 2$, then (2) and (4) are excluded. Th. IV.2: If $\Phi(\omega, t)\in\bigcup_{s=1}^{\infty}\mathscr{X}_s$, then it has a Poissonian local characteristic if and only if it is the first derivative of a composed Poisson process in $\mathscr{M}$ (the local characteristic is then explicitly computed). In Th. IV.3, for a g.s.p. Φ in $\bigcup_{s=1}^{\infty}\mathscr{X}_s$ having local characteristic, a non-negative integer k_0 can be obtained from the local characteristic, such that if $k_0=0$ the process is constant; if $k_0=1$, there is a process $f(\omega, t)$ in $\mathscr{X}\cap\mathscr{M}$ whose k_0th derivative is Φ, and the local characteristic of Φ determines all the moments of all the increments $\Delta_{s_1}\Delta_{s_2}f(\omega, t)$; if $k_0=1$, the local characteristic also determines all moments of $\Delta_h f(\omega, t)-E\{\Delta_h f(\omega, t)\}$. However, an example due to Stieltjes is given, of processes f_1, f_2 in $\mathscr{X}\cap\mathscr{M}$ whose first derivatives have the same local characteristics, but for which $\Delta_h f_j(\omega, t)-E\{\Delta_h f_j(\omega, t)\}$ do not have the same distributions for $j=1, 2$.

J. Feldman (Berkeley, Calif.)

6036:

Urbanik, K. The conditional expectations and the ergodic theorem for strictly stationary generalized stochastic processes. Studia Math. **17** (1958), 267–283.

Let $F(\omega, t)$ be a continuous stochastic process, whose conditional expectation $\mathscr{E}\{F(\omega, t)|\mathscr{F}\}$ ($\mathscr{F}$ a σ-field of sets) exists and has a version which is locally integrable in t. Let $\Phi(\omega, t)$ be the g.s.p. [see review above] $d^kF(\omega, t)/dt^k$.

Then $E\{\Phi(\omega, t)|\mathscr{F}\}$ is defined as

$$\frac{d^k}{dt^k}\mathscr{E}\{F(\omega, t)|\mathscr{F}\}.$$

The definition can be shown to depend only on Φ, and not on k, F, and the version chosen.

Theorem 1: If $E\{\Phi(\omega, t)\}$ exists, then $E\{\Phi(\omega, t)|\mathscr{F}\}$ exists (i.e., k, F can be chosen as above). The operation $E\{\cdot|\mathscr{F}\}$ satisfies the formal properties of the usual conditional expectation, when these are appropriately restated. Further, it commutes with d/dt.

Let Φ be a strictly stationary g.s.p., i.e., a kth derivative of a continuous process F_k whose kth increments $\Delta_h^{(k)}F_k$ form strictly stationary processes. For such a k, let $\mathscr{F}_h^{(k)}$ be the invariant σ-field of $\Delta_h^{(k)}F_k$ (which depends only on k and h, not on F_k), and $\mathscr{F}_\Phi=\bigcap_k\bigcap_{0<h<1}\mathscr{F}_h^{(k)}$. Φ is called indecomposable if all sets of $\mathscr{F}_\Phi$ have measure 0 or 1. Theorem 2: Strictly stationary g.s.p. with independent values are indecomposable. Theorem 3: Let Φ be a strictly stationary g.s.p. for which $E\{\Phi(\omega, t)\}$ exists. Then

$$T^{-1}\int_t^{t+T}\Phi(\omega, u)du\to E\{\Phi(\omega, t)|\mathscr{F}_\Phi\}$$

when $T\to\infty$, and the conditional expectation $E\{\Phi(\omega, t)|\mathscr{F}\}$ is a random variable independent of t.

The techniques combine g.s.p. techniques such as those established in the first paper of this series [first reference, #6033 above], use of Birkhoff's ergodic theorem, and standard probabilistic techniques as in Doob's *Stochastic processes* [Wiley, New York, 1953; MR 15, 445].

J. Feldman (Berkeley, Calif.)

6037:

★Schwartz, Laurent. La fonction aléatoire du mouvement brownien. Séminaire Bourbaki; 10e année: 1957/1958. Textes des conférences; Exposés 152 à 168; 2e éd. corrigée, Exposé 161, 23 pp. Secrétariat mathématique, Paris, 1958. 189 pp. (mimeographed)

The present paper contains a treatment of the basic notions of probability theory in connection with abstract spaces and Brownian notion. Several definitions concerning structure and convergence of random variables are introduced. Further, necessary and sufficient conditions of "similarity" and independence are given for two random variables ξ_1 and ξ_2 defined on the same topological space X in terms of the random variables $\langle\alpha', \xi_1\rangle$ and $\langle\alpha', \xi_2\rangle$ generated by $\alpha'\in X'$, where X' is the adjoint space of X. Gaussian and generalized random variables of second order are also defined and investigated in order to pass to the Wiener-Lévy random function (Brownian motion). It is shown that up to a similitude there is a unique normalized Wiener-Lévy function. Moreover, the normalized Wiener-Lévy function has the following properties: (1) $\xi(0)=0$, $\mathscr{E}(\xi(1))=1$, $\mathscr{E}(\xi^2(1))=1$; (2) it is a function with independent increments; (3) for any $\tau\geq 0$, the generalized random functions $t\to\xi(t+\tau)-\xi(\tau)$ are all similar; (4) for any $s>0$, the random functions $t\to\xi(ts)/\sqrt{s}$ are all similar. Conversely, the normalized Wiener-Lévy function is up to a similitude the unique generalized continuous random function of second order satisfying the above-mentioned properties which do not involve normality and normalization. Finally, the Wiener-Lévy random measure is considered. The theory is illustrated by many instructive examples.

R. Theodorescu (Bucharest)

6038:

Burke, Paul J. Equilibrium delay distribution for one channel with constant holding time, Poisson input and random service. Bell System Tech. J. 38 (1959), 1021–1031.

The author considers the queueing system $M/D/1$ when the next person to be served is drawn at random from those who are waiting. Previous work on waiting times for the "random service" problem has been confined to the queueing system $M/M/s$ [E. Vaulot, C. R. Acad. Sci. Paris **222** (1946), 268–269; MR 7, 461; F. Pollaczek, ibid., 353–355; MR 7, 461; J. Riordan, Bell System Tech. J. **32** (1953), 100–119; MR 14, 664; R. I. Wilkinson, ibid., 360–383]. This paper contains a number of charts illustrating the author's results (the delay distribution for various occupancy values, comparison of random and queued service with constant holding times, comparison of constant and exponential holding times for random service).

D. G. Kendall (Oxford)

6039:

Urbanik, K. An effective example of a Gaussian function. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. 7 (1959), 343–349. (Russian summary, unbound insert)

If I is the interval $[a, b)$, $f^*(I)$ is $f(b)-f(a)$, and $I+t$ is the interval I translated through t, the author calls a real-valued function f on the positive half-line a Gaussian function if the following conditions are satisfied. The relative measure of the t-set with $f^*(I+t)<x$ is to exist and be the probability that a Gaussian normalized random variable lies in I; the relative measure of the t-set with $f^*(I_j+t)<x_j$, $j\leq n$, is to be the product of the individual values for each j, whenever the indicated intervals are disjoint. The author has proved [Studia Math. **17** (1958), 335–348; MR **21** #3047] that almost every sample function of a normalized separable Brownian motion process is a Gaussian function. In this paper he actually constructs a Gaussian function.

J. L. Doob (Urbana, Ill.)

6040:

Pyke, Ronald. The supremum and infimum of the Poisson process. Ann. Math. Statist. **30** (1959), 568–576.

Let $\{X(t);\ t\geq 0\}$ be a separable Poisson process with shift, i.e., such that

$$\log E(e^{i\omega X(t)}) = -it\omega\alpha+\lambda t(e^{i\omega}-1),$$

for all real ω, and $\alpha>0$, $\lambda>0$. The author obtains explicit expressions for $P[\sup_{0\leq t\leq T}X(t)\leq x]$ and for $P[\inf_{0\leq t\leq T}(Xt)\leq x]$. Applications to queuing theory are discussed.

Z. W. Birnbaum (Seattle, Wash.)

6041:

Samuels, J. Clifton; and Eringen, A. Cemal. On stochastic linear systems. J. Math. Phys. **38** (1959/60), 83–103.

"The present paper is concerned with the mathematical treatment of systems governed by nth order linear differential equations having random coefficients. The authors study (i) differential equations containing small randomly varying parameters, (ii) equations containing slowly varying random coefficients, (iii) equations containing only one random coefficient. The methods developed are afterwards applied to an RLC circuit with random capacity

variations and to dynamic instability of an elastic bar subject to randomly varying time-dependent axial force. The theory predicts a phenomenon of instability in a mean square sense which has also been observed experimentally. This is discussed at the end of the paper." (From the author's introduction)

R. A. Silverman (New York, N.Y.)

STATISTICS

See also 6183.

6042:

★**Dugué, Daniel. Traité de statistique théorique et appliquée: analyse aléatoire, algèbre aléatoire.** Collection d'ouvrages de mathématiques à l'usage des physiciens, Masson et Cie, Paris, 1958. xiv+313 pp. 5200 fr.

This book is really one of selected topics in probability and statistics; the selection seems to have been made from a rather personal point of view. The first part of the book develops probability and characteristic function techniques which are used to prove weak and strong laws of large numbers and the law of the iterated logarithm. There is a thick slice of derivations of the limiting distributions of one- and two-sided Kolmogorov-Smirnov statistics, and a 14-page chapter on the continuity of random functions culminating in a stochastic version of the Weierstrass approximation theorem. The small attempt at statistical theory is concentrated in a short Fisherian chapter on the subjects of maximum likelihood, information and efficiency. The above is all in the first part of the book which is called "Analyse aléatoire". The second part, "Algèbre aléatoire", starts with a short chapter on characterization of some distribution laws. Then follows the derivation of various distributions related to the normal law (t, F, chi-square, etc.). Finally there is considerable treatment of topics in the analysis and design of experiments: some elementary analysis of variance models, Latin squares and what they have to do with Galois fields, balanced incomplete blocks, projective geometries, finite geometries, confounding and interaction. There is, of course, a table of contents, but a sorely needed index is missing. The system of giving references leaves a little to be desired. Many important theorems are not ascribed to anyone; among those that are, the reference is often incomplete, being in name only, with the relevant article not appearing in the bibliography. The central limit theorem is produced in passing in a few lines in the section on the law of the iterated logarithm. There is essentially nothing on confidence procedures or tests of hypotheses.

M. Dwass (Evanston, Ill.)

6043:

★**Whitney, D. Ransom. Elements of mathematical statistics.** Henry Holt and Co., New York, 1959. ix+148 pp. $4.75.

This book is intended for the undergraduate who has had little more than calculus. The preliminaries are on empirically plausible sample statistics and on combinatorics. There is a slight nod toward the axiomatics of probability theory, and random variables are informally introduced as primitive objects. Surprisingly, the characteristic function is introduced instead of the moment generating function which usually appears at this level. Statistical procedures first appear in a short expository chapter on the 2-sided, one population Kolmogorov-Smirnov test and the chi-square goodness of fit test. The book proceeds to the methods of moments and maximum likelihood for estimation and then (in the same chapter) to joint distributions, some treatment of the distribution of $\bar{x}$, t, s^2, the central limit theorem, and some remarks on confidence intervals. The last three chapters are labeled "Testing a hypothesis", "Trend" (one-dimensional regression with normal variables and the Wilcoxon non-parametric version), and "Comparison of two groups" (comparison of means and variances with normal variables and the Wilcoxon-Mann-Whitney test). The book is extremely heuristic, casual, and short, but the style and format are attractive and there are many problems.

M. Dwass (Evanston, Ill.)

6044:

Askovitz, S. I. Graphic methods based upon properties of advancing centroids. J. Amer. Statist. Assoc. **54** (1959), 668–673.

"Short-cut techniques involving centroids of sets of points are extended to least squares applications in line fitting and to determining the mean of a frequency distribution." *Author's summary*

6045:

Carlson, Phillip G. A recurrence formula for the mean range for odd sample sizes. Skand. Aktuarietidskr. **1958**, 55–56 (1959).

Let $E\omega_k$ be the mean range of a sample of size k taken from a given population. Following a suggestion by E. J. Gumbel, the author uses Tippett's formula for $E\omega_k$ together with a recurrence formula due to Romanowsky to obtain a recursion formula for $E\omega_{2n+1}$ in terms of $E\omega_{n+i+1}$ for i ranging from 0 to $n-1$. There are several typographical errors in the paper. The recursion formula should read:

$$E\omega_{2n+1} = (2n+1)\sum_{i=0}^{n-1}\binom{n}{i}\frac{(-1)^{n+i+1}}{n+i+1}E\omega_{n+i+1}.$$

H. Raiffa (Cambridge, Mass.)

6046:

Ångström, Knut H. An asymptotic expansion of bias in a non-linear function of a set of unbiased characteristics from a finite sample. Skand. Aktuarietidskr. **1958**, 40–46 (1959).

Suppose x_n, $n=1, 2, \cdots$, is a sequence of random variables converging in probability to a constant m as $n\to\infty$. If $f(x)$ and its first two derivatives $f'(x)$, $f''(x)$, are continuous in some neighborhood of $x=m$, if $f(x_n)$ converges in probability to $f(m)$ and if $\varepsilon(x_n-m)^2$ converges to zero as $n\to\infty$, then it is shown that the asymptotic expression

$$\varepsilon(f(x_n)) \doteq f(m)+\tfrac{1}{2}f''(m)\varepsilon(x_n-m)^2$$

holds as $n\to\infty$. This result is extended to the case in which x_n is a vector. *S. S. Wilks* (Princeton, N.J.)

6047:

Dempster, A. P. Generalized D_n^+ statistics. Ann. Math. Statist. **30** (1959), 593–597.

Let $F_n(x)$ be the sample c.d.f. determined by a sample

of size n of a random variable X which has uniform distribution on (0, 1). For $0 \leq \delta < 1$ and $0 < \varepsilon < 1$ the line joining $(0, \delta)$ and $(1-\varepsilon, 1)$ is called the (δ, ε)-barrier. The author obtains an explicit expression for the probability that $F_n(x)$ does not cross a (δ, ε)-barrier. He also considers the class of $(0, d)$-barriers, i.e., of lines through the origin and $(1-d, 1)$, and defines for a given sample a pair of statistics (U^*, i^*) by rotating the $(0, d)$-barrier until it touches the graph of $F_n(x)$ and calling the coordinates of the point of contact (U^*, i^*). The joint distribution of (U^*, i^*) and the marginal distributions are then obtained.

Z. W. Birnbaum (Seattle, Wash.)

6048:

Rybarz, J. Ein einfacher Beweis für das dem X^2-Verfahren zugrundeliegende Theorem. Metrika **2** (1959), 89–93.

The author shows that the multinomial distribution is a conditional distribution of Poisson variates subject to a fixed total. He then uses the central limit theorem for the Poisson distribution to derive the asymptotic normal distribution. The statistic of interest is the exponent of the normal distribution and the desired conclusions follow therefrom.

S. Kullback (Washington, D.C.)

6049:

Borenius, Gustaf. On the distribution of the extreme values in a sample from a normal distribution. Skand. Aktuarietidskr. **1958**, 131–166 (1959).

$\bar{x}$ is the mean and s the root-mean-square deviation of a sample of n independent values from a normal population. The relative deviation of a sample value x_i is $r_i = (x_i - \bar{x})/s$. Explicit expressions are found for the density function of the distribution of r_n, the relative deviation of the maximum value in the sample, for $n = 3, 4, 5, 6$; and frequency curves are drawn. With the aid of a recursion formula, frequency curves are also drawn for $n = 8, 10, 12, 16, 20, 30$. Explicit expressions are found, and curves are drawn, for the distribution of $\max_i |r_i|$ when $n = 3, 4, 5$, and the upper part of the distribution when $n = 6$ is also given. Elementary geometric methods are used, different from those of F. E. Grubbs [Ann. Math. Statist. **21** (1950), 27–58; MR **11**, 527].

F. J. Anscombe (Chicago, Ill.)

6050:

Kamat, A. R. Hypergeometric expansions for incomplete moments of the bivariate normal distribution. Sankhyā **20** (1958), 317–320.

Hypergeometric expansions are found for incomplete moments

$$[m, n] \equiv \int_0^\infty \int_0^\infty x^m y^n p(x, y)\,dx\,dy$$
$$= 2^{\frac{1}{2}(m+n-4)}\pi^{-1}\left\{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right) \times F\left(-\frac{m}{2}, -\frac{n}{2}, \frac{1}{2}; \rho^2\right) + 2\rho\Gamma\left(\frac{m+2}{2}\right)\Gamma\left(\frac{n+2}{2}\right)F\left(-\frac{m-1}{2}, -\frac{n-1}{2}, \frac{3}{2}; \rho^2\right)\right\}$$

and absolute moments

$$(m, n) \equiv \int_{-\infty}^\infty \int_{-\infty}^\infty x^m y^n p(x, y)\,dx\,dy$$
$$= 2^{\frac{1}{2}(m+n)}\pi^{-1}\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)F\left(-\frac{m}{2}, -\frac{n}{2}, \frac{1}{2}; \rho^2\right)$$

of the bivariate normal distribution

$$p(x, y) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}\exp\left[-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)\right].$$

For the purpose of solving truncation problems numerical values of $[m, n]$ for $m+n \leq 4$ and $\rho = -0.9(0.1)1.0$ are given.

T. Kitagawa (Fukuoka)

6051:

Adke, S. R. Generalized affinity and a class of distance functions. Proc. Nat. Inst. Sci. India. Part A **25** (1959), 104–110.

A generalization of the "affinity" of Bhattacharya [Bull. Calcutta Math. Soc. **35** (1943), 99–109; MR **6**, 7] is introduced and properties akin to those considered by Adhikari and Joshi [Publ. Inst. Statist. Univ. Paris **5** (1956), 57–74; MR **19**, 329] are investigated.

H. Teicher (Lafayette, Ind.)

6052:

Kamat, A. R. Incomplete moments of the trivariate normal distribution. Sankhyā **20** (1958), 321–322.

Expressions for the absolute moments of the trivariate normal distribution

$$[l, m, n] = \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty |x|^l |y|^m |z|^n\, p(x, y, z)\,dx\,dy\,dz,$$

where $p(x, y, z)$ is the trivariate normal probability density with mean values $E(x) = E(y) = E(z) = 0$, and unit variances $\sigma^2\{X\} = \sigma^2\{Y\} = \sigma^2\{Z\} = 1$ and with correlation coefficient $\rho_{12}, \rho_{13}, \rho_{23}$ are given up to weight four, $l+m+n \leq 4$, and their use in some problems of truncation is indicated.

T. Kitagawa (Fukuoka)

6053:

Maxwell, A. E. Maximum likelihood estimates of item parameters using the logistic function. Psychometrika **24** (1959), 221–227.

The problem considered here reduces to that of estimating α and β by the method of maximum likelihood from the likelihood function

$$\prod_{i=1}^{k} \binom{n_i}{r_i} P_i^{r_i}(1-P_i)^{n_i-r_i},$$

where $r_1, \cdots, r_k$ are independent random variables having the binomial distributions indicated, and $P_i = [1 + e^{-\alpha-\beta x_i}]^{-1}$, the x_i being fixed numbers. This problem as one in statistical estimation theory, as pointed out by the author, has been considered by Anscombe, Berkson, Birnbaum and Hodges. The main point of this paper is to show how to apply the theory to the estimation of parameters of test items.

S. S. Wilks (Princeton, N.J.)

6054:

Elfving, G.; Sitgreaves, R.; and Solomon, H. Item

selection procedures for item variables with a known factor structure. Psychometrika **24** (1959), 189–205.

This paper deals with a problem in psychometrics which may be mathematically formulated as follows. $x_1, \cdots, x_N$ are random variables such that

$$x_i = \sum_{j=1}^{k} a_{ij}y_j + \varepsilon_i \quad (i=1, \cdots, N),$$

$$z = \sum_{j=1}^{k} c_j y_j + \eta,$$

where $\varepsilon_1, \cdots, \varepsilon_N$ are random variables having zero means, and non-singular covariance matrix V, η is a random variable having zero mean, variance σ^2 and zero intercorrelations with the ε_i, while two alternative assumptions are made concerning $y_i, \cdots, y_k$: (I) the y_i are unknown constants; (II) the y_i are random variables with zero means and known non-singular covariance matrix T. Denoting $\|a_{ij}\|$ by A (assumed non-singular), $(c_1, \cdots, c_k)$ by c', and $(x_1, \cdots, x_N)$ by x, and using only linear unbiased minimum-variance estimators for z the authors show that under (I) the estimator for z is given by $c'(A'V^{-1}A)^{-1}A'V^{-1}x$ and the variance of this estimator is $c'(A'V^{-1}A)c+\sigma^2$. Under (II) the estimator for z is $c'(T^{-1}+A'V^{-1}A)^{-1}A'V^{-1}x$ and the estimator of this estimator is $c'(T^{-1}+A'V^{-1}A)^{-1}c+\sigma^2$. The theory is illustrated with several simple numerical examples. *S. S. Wilks* (Princeton, N.J.)

6055:

Dwass, Meyer. Multiple confidence procedures. Ann. Inst. Statist. Math. Tokyo. **10** (1959), 277–282.

Suppose $x_1, \cdots, x_n$ is a set of independent variables having normal distributions $N(\mu_1, \sigma^2), \cdots, N(\mu_n, \sigma^2)$ and let s^2 be a random variable independent of $x_1, \cdots, x_n$ such that ks^2/σ^2 has a chi-square distribution with k degrees of freedom. Let $\bar{x}=n^{-1}\sum_i x_i$ and T_p be defined by

$$P[(\sum_i |x_i-\bar{x}|^p)/k^{1/2}s > T^p] = \alpha$$

where $1 \leq p < \infty$ and $0<\alpha<1$. It is shown by using Hölder's inequality that the probability is $1-\alpha$ that

$$|\sum_i a_i\mu_i - \sum_i a_i x_i| \leq [\sum_i (a_i-\hat{a})^q]^{1/q}T_p$$

for all (real) vectors $(a_1, \cdots, a_n)$ satisfying $\sum_i a_i = 0$, where $1/p+1/q=1$, and where $\hat{a}$ satisfies

$$\sum_i |a_i-\hat{a}|^q = \inf_c \sum_i |a_i-c|^q.$$

Similarly, it is shown that the probability is $1-\alpha$ that

$$|\sum_i a_i\mu_i - \sum_i a_i x_i| \leq (\sum_i a_i^q)^{1/q}\, T_p'$$

for all real vectors $(a_1, \cdots, a_n)$ satisfying $\sum_i a_i=0$, where T_p' is defined by

$$P[(\sum_i |x_i-\hat{x}|^p)/k^{1/2}s > T_p'] = \alpha \quad \text{and}$$

$$\sum_i |x_i-\hat{x}|^p = \inf_c \sum_i |x_i-c|^p.$$

The author shows how various results on simultaneous confidence intervals previously obtained by Tukey, Scheffé, and by Bose and Roy reduce to special cases of these results. *S. S. Wilks* (Princeton, N.J.)

6056:

Bahadur, R. R. Examples of inconsistency of maximum likelihood estimates. Sankhyā **20** (1958), 207–210.

6057:

Quensel, C.-E. Some sampling problems when a stratification variable follows a logarithmic normal distribution. Skand. Aktuarietidskr. **1958**, 177–184 (1959).

Sampling problems when a stratification variable follows a logarithmic normal distribution are discussed in this paper. The population is divided into k strata with the stratification points $x_0(=0)$, $x_1, x_2, \cdots, x_k\ (=\infty)$, and the mean μ_i and the standard deviation σ_i in these strata are calculated. In view of the optimal stratification point in the case of standardized normal distributions, the optimal stratification points are approximately given. *T. Kitagawa* (Fukuoka)

6058:

Block, Eskil. Numerical considerations for the stratification of variables following a logarithmic normal distribution. Skand. Aktuarietidskr. **1958**, 185–200 (1959).

In order to realize the aim of stratified sampling in reducing the variance of estimation of the mean of a population, the paper discusses the decision of how to fix the stratification points when the stratification variable follows a logarithmic normal distribution. The optimum stratification values are given for the cases when the number of strata are 2, 3 or 4, to the effect that

$$\log x_i = l_1+\sigma_1(H_i+\sqrt{\sigma_2})$$

where the author assumes that

$$f(x, y) = \frac{1}{\sigma_1\sigma_2 2\pi(1-r^2)^{1/2}xy}\exp -\frac{1}{2(1-r^2)}\left\{\left(\frac{\log x-l_1}{\sigma_1}\right)^2 -2r\frac{\log x-l_1}{\sigma_1}\frac{\log y-l_2}{\sigma_2}+\left(\frac{\log y-l_2}{\sigma_2}\right)^2\right\}$$

and $H=0$ for two strata; $H_1=-0.63$, $H_2=0.63$ for three strata; $H_1=-1.00$, $H_2=0.00$, $H_3=1.00$ for four strata.

The results are valid only for reasonably strong correlation and for low sampling rates, but some advice is also given for other cases. *T. Kitagawa* (Fukuoka)

6059:

Rao, C. Radhakrishna. Maximum likelihood estimation for the multinomial distribution with infinite number of cells. Sankhyā **20** (1958), 211–218.

The author considers a multinomial distribution with an infinite number of cells with frequencies $\pi_1^0, \pi_2^0, \cdots$ for which there are observed frequencies $p_1, p_2, \cdots$. He defines the maximum likelihood estimate of the π_i's belonging to an admissible class A of distributions, if it exists, as the distribution which maximizes $\sum p_i \log \pi_i$ when $(\pi_1, \pi_2, \cdots)$ is restricted to A. He establishes the consistency of this estimate under the hypothesis that $\sum \pi_i^0 \log \pi_i^0 > 0$. He also considers the case where $\pi_i(\theta)$ is a function of the parameter θ and examines the maximization of $\sum p_i \log \pi_i(\theta)$ and the roots of the corresponding maximum likelihood equation $\sum p_i/\pi_i \cdot d\pi_i/d\theta = 0$. *R. A. Leibler* (Princeton, N.J.)

6060:

Moran, P. A. P. The power of a cross-over test for the artificial stimulation of rain. Austral. J. Statist. **1** (1959), 47–52.

This paper compares the powers of two statistical tests associated with two experiment designs E_1 and E_2 for testing the effect of cloud seeding on precipitation. Two similar geographical areas A_1 and A_2 are considered. Let x and y refer to precipitation measurements on A_1 and A_2 respectively. In design E_1, precipitation measurements $(x_1, y_1), \cdots, (x_m, y_m)$ are obtained on A_1 and A_2 for m time periods, where cloud seeding has been applied to A_1 but not A_2, while precipitation measurements $(x_{m+1}, y_{m+1}), \cdots, (x_{m+n}, y_{m+n})$ are obtained on A_1 and A_2 for n time periods where cloud seeding was applied to neither A_1 nor A_2. E_2 differs from E only in that in E_2 the second set of measurements was obtained where cloud seeding was applied to A_2 but not A_1. The statistical tests developed for the two experiment designs, assuming (x, y) to have a bivariate normal distribution under each of the four different seeding (or non-seeding) conditions of A_1 and A_2, are both Student t-tests, which need not be given explicitly here. It is shown that the Student test for E_2 (the cross-over design) is more powerful than that for E_1.

S. S. Wilks (Princeton, N.J.)

6061:

Linhart, H. Critère de sélection pour le choix des variables dans l'analyse de régression. Schweiz. Z. Volkswirtschaft Statist. **94** (1958), 202–232.

From a sample of size n of a $(k+1)$-dimensional normal random variable, $(X_0, X_1, \cdots, X_k)$, it is required to predict either $E(X_0)$ or X_0 when only $(X_1, X_2, \cdots, X_k)$ is known. Let $l_{[r]}$ be the length of the confidence interval for $E(X_0)$ (or X_0 when that is considered more important) when a particular r predictor variates are omitted. The criterion proposed for deciding whether to omit them or not is based on $E(l_{[0]})/E(l_{[r]})$, where the expectations are taken over the distribution of the sample and predictor variables. This ratio is monotonically increasing with a population parameter $\gamma_{[r]}$. The paper establishes that a sample statistic $c_{[r]}$ is such that $(1-c_{[r]})^{1/2}$ has the distribution of a multiple correlation coefficient in a sample of size $n-k+r$ from an $(r+1)$-dimensional normal distribution having a population R of $(1-\gamma_{[r]})^{1/2}$. This provides a test of the hypothesis $E(l_{[0]})>E(l_{[r]})$ which, if not rejected, requires the omission of the r variates. The cases of prediction of $E(X_0)$ and X_0 are described in parallel and an application is made to psychological test data. Special tables are appended.

M. Stone (Cambridge, England)

6062:

Madansky, Albert. Bounds on the expectation of a convex function of a multivariate random variable. Ann. Math. Statist. **30** (1959), 743–746.

Suppose X is an r-dimensional vector-valued random variable with cdf $\psi(x)$. The ith moment of $\psi(x)$ with respect to the set of continuous functions $\{f_i(x)\}$, $i=1, 2, \cdots, n$, is defined to be $\mu_i(\psi)=\int_I f_i(x)d\psi(x)$, where I is the bounded r-dimensional rectangle over which $\psi(x)$ is defined. Also, the nth moment space M_n with respect to $\{f_i(x)\}$ is defined as the set of all points $(\mu_1, \cdots, \mu_n)$ in n-dimensional Euclidean space whose coordinates are the moments $\mu_1(\psi), \cdots, \mu_n(\psi)$ with respect to $\{f_i(x)\}$ for some cdf $\psi(x)$. Let $g(x)$ be some continuous convex function of the random vector variable X and let C_{r+1} be the surface in $(r+1)$-dimensional Euclidean space defined by $z_1=x_1$, $z_2=x_2, \cdots, z_r=x_r, z_{r+1}=g(x)$. The author determines the boundary of the convex hull of C_{r+1} and, using it, obtains inequalities on the expected value $Eg(X)$ in terms of $g(EX)$ and $\{EX_i, i=1, \cdots, r\}$. Finally, if the elements of the random vector X are mutually independent, the paper presents a sharper inequality, viz, a closer upper bound on $Eg(X)$.

R. Gnanadesikan (Murray Hill, N.J.)

6063:

Pratt, John W. Remarks on zeros and ties in the Wilcoxon signed rank procedures. J. Amer. Statist. Assoc. **54** (1959), 655–667.

Using the Wilcoxon signed rank procedures with zeros neglected a sample judged to be not significantly positive can become significantly positive if the zeros are included and all of the observations are reduced slightly. Rational methods are given to prevent this and related paradoxes.

I. R. Savage (Minneapolis, Minn.)

6064:

Sarmanov, O. V.; and Vistelius, A. B. Correlation between percentage magnitudes. Dokl. Akad. Nauk SSSR **126** (1959), 22–25. (Russian)

Let $x_1, \cdots, x_n; z_1, \cdots, z_m$ be positive random variables. The random variables $\xi_i=100x_i/(x_1+\cdots+x_n+z_1+\cdots+z_m)$, $i=1, \cdots, n$ and $\zeta_j=100z_j/(x_1+\cdots+x_n+z_1+\cdots+z_m)$, $j=1, \cdots, m$, are called percentage magnitudes by the authors. Suppose that z_1 and z_2 are independent and that (z_1, z_2) is independent of (x_1, x_2). Elementary computations show that the absolute value of the correlation coefficient of ξ_1/ζ_1 and ξ_2/ζ_2 is always smaller than or equal to the absolute value of the correlation coefficient of x_1 and x_2. Moreover they have always the same sign and vanish together.

L. Schmetterer (Hamburg)

6065:

Hooper, J. W.; and Theil, H. The extension of Wald's method of fitting straight lines to multiple regression. Rev. Inst. Internat. Statist. **26** (1958), 37–47.

Given the regression model

$$z_t = \alpha+\beta x_t+\gamma y_t+u_t, \tag{1}$$

where the $\{x_t, y_t\}$ are fixed and the disturbances $\{u_t\}$ are uncorrelated with zero mean and variance σ^2. The authors compare a new four-group method of estimating the regression coefficients with a nine-group method previously proposed by Gibson and Jowett [Appl. Statist. **6** (1957), 189–197]. These methods are extensions of methods previously proposed by Wald and Bartlett for a single independent variable (x) in which the observations are divided into two or three groups according to increasing values of x and the slope determined from a straight line through the centers of gravity of the two outer groups. The nine-group method is based on ordering of both the x's and y's. The same ordering is done for the four-group method, in which three mutually exclusive groups on the outside are used to estimate the three parameters with a fourth interior group not used.

The two methods of estimating the parameters in (1) are compared on the basis of the generalized variance for

$b(=\hat{\beta})$ and $c=(\hat{\gamma})$, $V=\mathrm{Var}(b)\mathrm{Var}(c)-\mathrm{Cov}^2(b,\ c)$. These comparisons are made for these cases: (i) $\{x_i,\ y_i\}$ are uniformly distributed over an equilateral triangle and (ii) $\{x_i,\ y_i\}$ are uniformly distributed over the unit square. In both cases, the four-group method is less than 90% as efficient as the nine-group method.

This reviewer feels that the nine-group designation is somewhat confusing, because the estimation is based on dividing first according to x to obtain b_{yx} and b_{zx} and then according to y to obtain b_{zy} and b_{xy}. These four b's are then used to obtain the partial regression coefficients, e.g.,

$$c = b_{zy.x} = \frac{b_{zy}-b_{zx}b_{xy}}{1-b_{xy}b_{yx}}.$$

Hence, efficiencies of the two methods are actually based on three vs. four groups. *R. L. Anderson* (Raleigh, N.C.)

6066:

Sobel, Milton; and Groll, Phyllis A. Group testing to eliminate efficiently all defectives in a binomial sample. Bell System Tech. J. **38** (1959), 1179–1252.

The group testing problem is the following. Suppose that N individuals are at hand from a population in which the probability of possessing a certain defect is $p\ (=1-q)$ and that it is desired to identify all the individuals in the group possessing that defect. Suppose also that a "group testing" technique is available which, when applied to a group of x individuals, tells whether there are any defective individuals in the group. Then the problem is to design a program of group tests that minimizes the expected number of tests needed to identify all the defective individuals.

At any stage of the procedure let n be the number of individuals whose defectiveness has not been ascertained, and m be the number that have been in batches subjected to group tests that revealed the presence of defectives, so that $n-m$ is the number that never have been tested. Let $G_1(m,\ n)$ denote the expected number of tests remaining to be performed. At the outset $m=0$, $n=N$, and if x is the size of the group subjected to the first test

$$G_1(0,\ N) = 1+(q^xG_1(0,\ N-x)+(1-q^x)G_1(x,\ N)).$$

The size of the first test-group, x, is chosen to minimize this expression. The function $G_1(m,\ n)$ is calculated from the recursion formula

$$G_1(m,\ n) = 1+\min_{1\leq x\leq m-1}(f(x,\ m)G_1(m-x,\ n-x)$$
$$+(1-f(x,\ m))G_1(x,\ n))$$

where $f(x,\ m)=(q^x-q^m)/(1-q^m)$, because, it can be proved, if a group of m is known to contain defective individuals and if a group test on a subgroup of x reveals the presence of defective individuals, then the probability distribution of the number of defectives in the $m-x$ individuals not in the subgroup is the same as if no tests had ever been made on them. Thus these individuals can be pooled with the $n-m$ individuals that have never been tested.

Extensive tables of optimal test group size and related functions are presented. This procedure, while not entirely optimal, is shown to be superior to the two group testing procedures previously suggested in the literature.

R. Dorfman (Cambridge, Mass.)

6067:

Linhart, H. Techniques for discriminant analysis with discrete variables. Metrika **2** (1959), 138–149.

A discrete p-variate space is mapped into I regions R_j $(j=1,\ 2,\ \cdots,\ I)$ and the probability of any specified p-tuple being in R_j is supposed known or capable of estimation from observational frequencies. Given c_{ij}, the cost of misclassification of a p-tuple $k,\ l,\ m,\ n,\ \cdots$ from i into a region j, the optimum allocation for $k,\ l,\ m,\ n,\ \cdots$ is said to be j_0 if the expected cost of misclassification into j_0 (taken over all $i=1,\ 2,\ \cdots,\ I$) ir a minimum $(j=1,\ 2,\ \cdots,\ I)$. The author shows how to apply this concept numerically and generalizes it to the case where some of the variates are continuous.

H. L. Seal (New Haven, Conn.)

6068:

Jackson, J. Edward; Freund, Richard A.; and Howe, William G. Errors associated with process adjustments. Virginia J. Sci. **10** (1959), 3–26.

The authors discuss the effects of correcting an industrial process as a function of a sample. An investigation is made of the probabilities of making an undercorrection, a helpful overcorrection, a harmful overcorrection, and a correction in the wrong direction. Several adjustment rules are analyzed and the operating characteristic curves given. {Errata: Caption, p. 7 should read "Figure 3. Probability curves for Case 2"; caption, p. 9 should read "Figure 2. Probability curves for Case 1"; p. 13 and 14, change 3.0 to 1.5 in caption; Table 1, columns labeled "lower limit" and "upper limit" are reversed.}

M. Zelen (Washington, D.C.)

6069:

Davies, O. L. Some statistical aspects of the economics of analytical testing. Technometrics **1** (1959), 49–61.

Two situations are described in which the cost of analytical testing of the chemical constitution of a product must be weighed against losses resulting from inaccurate grading. The first case relates to the sale of finished product, and one of the loss components is loss of good will when substandard material is sold. Rather than assess the value of good will, the author arbitrarily fixes one point on the operating characteristic, after the manner of G. Horsnell [J. Roy. Statist. Soc. Ser. A **120** (1957), 148–201]. The second case relates to an intermediate product, for which it is supposed that all losses can be assessed.

F. J. Anscombe (Chicago, Ill.)

6070:

Mickey, M. R. Some finite population unbiased ratio and regression estimators. J, Amer. Statist. Assoc. **54** (1959), 594–612.

A class of unbiased ratio and regression type estimators are generated for random sampling, without replacement, from a finite population, $(y_\beta,\ x_{1\beta},\ \cdots,\ x_{p\beta})$, $\beta=1,\ \cdots,\ n$. The construction is based on the fact that $\bar{y}-\sum a_i(\bar{x}_i-\mu_{x_i})$ is an unbiased estimator of μ_y, for any set of constants $a_1,\ \cdots,\ a_p$. The constants are chosen as functions of the first α observations in the sample of size N. An averaging then yields an unbiased estimator with smaller variance. Some examples and a discussion of the estimator variance are also given. *I. Olkin* (East Lansing, Mich.)

6071:

Freiberger, Walter; and Grenander, Ulf. Approximate distributions of noise power measurements. Quart. Appl. Math. 17 (1959), 271–283.

"The frequency functions of certain spectral estimates for stationary normal processes are studied analytically and numerically. An approximation is obtained for the case of a Poisson weight function and compared to the true distribution. The eigenvalues of products of Toeplitz matrices play a crucial role in the sampling theory of quadratic forms; an approximation to their distribution is discussed and its accuracy is studied numerically. This leads to approximate probability densities thought to be valid for moderate or even small sample sizes." (Authors' summary) *M. Rosenblatt* (Providence, R.I.)

6072:

Anderson, T. W. On asymptotic distributions of estimates of parameters of stochastic difference equations. Ann. Math. Statist. **30** (1959), 676–687.

The author considers the autoregression $x_t = \alpha x_{t-1} + u_t$, the u_t being for the most part supposed independently and identically distributed. He obtains several results, of which the following are typical: If

$$\hat{\alpha} = \frac{\sum_1^T x_t x_{t-1}}{\sum_1^T x_{t-1}^2}$$

then $\sqrt{(T)}(\hat{\alpha}-\alpha)$ is asymptotically $N(0, 1-\alpha^2)$ provided $|\alpha| < 1$ and the u's have finite variance. If $|\alpha| > 1$ then $[\alpha^T/(\alpha^2-1)](\hat{\alpha}-\alpha)$ will under certain conditions have a limiting distribution. If $|\alpha| > 1$ and the u's are $\mathrm{NID}(0, \sigma^2)$ then $(\sum x_{t-1}^2)^{1/2}(\hat{\alpha}-\alpha)$ is asymptotically $N(0, \sigma^2)$. The vector case is also considered.

P. Whittle (Cambridge, England)

6073:

White, John S. The limiting distribution of the serial correlation coefficient in the explosive case. II. Ann. Math. Statist. **30** (1959), 831–834.

Using the results of an earlier paper [same Ann. **29** (1958), 1188–1197; MR **20** #7377] the author shows that for any α, with the possible exception of the cases $|\alpha| = 1$, the quantity

$$W = \frac{\hat{\alpha}-\alpha}{\hat{\sigma}}\left(\sum_1^T x_{t-1}^2\right)^{1/2}$$

is distributed asymptotically $N(0, 1)$. Here $\hat{\alpha}$ and $\hat{\sigma}$ are the maximum likelihood estimates of the parameters of the autoregression $x_t = \alpha x_{t-1} + u_t$, the u's being $\mathrm{NID}(0, \sigma^2)$.

P. Whittle (Cambridge, England)

6074:

Moran, P. A. P. Random processes in economic theory and analysis. Sankhyā **21** (1959), 99–126.

A survey article, restricted more to statistical (as opposed to econometric) literature of the English school than the title suggests. To this special literature it provides a clear and valuable guide.

P. Whittle (Cambridge, England)

NUMERICAL METHODS

See also 5544, 5753, 5754, 6044, 6164.

6075:

★Collatz, L. Fixed-point theorems and monotonic operators for initial-value problems. Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 56–63. Libreria Eredi Virgilio Veschi, Rome, 1959. xii + 158 pp.

Bei der numerischen Behandlung von Anfangswertaufgaben mit Rechenanlagen sind Fehlerabschätzungen besonders wichtig. Solche Abschätzungen sind noch nicht so weit entwickelt, wie es für die Anwendungen erforderlich wäre. Man hat jedoch—hauptsächlich mit funktionalanalytischen Methoden—bereits eine Reihe von brauchbaren Ergebnissen erhalten. Ueber diese wird berichtet. Es handelt sich um Fehlerabschätzungen für das Iterationsverfahren $u_{n+1} = Tu_n$ in metrischen und pseudometrischen Räumen und solche Verfahren mit monotonem Operator. Ferner werden Abschätzungen mit Hilfe der Theorie von Leray-Schauder und dem Schauder'schen Fixpunktsatz erwähnt. Numerische Beispiele.

J. Schröder (Hamburg)

6076:

Samelson, Klaus. Faktorisierung von Polynomen durch funktionale Iteration. Bayer. Akad. Wiss. Math.-Nat. Kl. Abh. **95** (1959), 26 pp.

After a brief, but informative, summary of known iterative methods of factorization, a new method is presented with second-order convergence, which, however, requires the inversion at each step of the matrix of the resultant of the current approximations to the factors. Special attention is given to the case where one of the factors is linear. The paper is of interest, not only for the new method, but for the light it throws on iterative methods in general.

A. S. Householder (Oak Ridge, Tenn.)

6077:

Gatteschi, Luigi. Formule asintotiche "ritoccate" per le funzioni di Bessel. Tabulazione e grafici delle funzioni ausiliarie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 506–514.

The author examines the use of the modulus $\Psi_\nu(x)$ and phase $\Phi_\nu(x)$ for tabulating the Bessel functions

$$J_\nu(x) = (2/\pi x)^{1/2}\Psi_\nu(x)\cos\Phi_\nu(x),$$
$$Y_\nu(x) = (2/\pi x)^{1/2}\Psi_\nu(x)\sin\Phi_\nu(x),$$

for large values of x. Some graphs and short tables are included for computing $\Psi_\nu(x)$ and $\Phi_\nu(x)$ when $-1 \leq \nu \leq 1$.

F. W. J. Olver (Teddington)

6078:

DiDonato, A. R.; and Hershey, A. V. New formulas for computing incomplete elliptic integrals of the first and second kind. J. Assoc. Comput. Mach. **6** (1959), 515–526.

Reihenentwicklungen für unvollständige elliptische Integrale, die sich eng an die hypergeometrischen Reihen der vollständigen Integrale anschließen. Ihre Verwendung als Unterprogramm für Rechenanlagen wird untersucht.

F. Stallmann (Braunschweig)

6079:

Suschowk, Dietrich. Explicit formulae for 25 of the associated Legendre functions of the second kind. Math. Tables Aids Comput. **13** (1959), 303–305.

The 25 functions in question are $Q_n^k(x)$ for $k=0(1)4$ and $n=0(1)4$. They have been calculated from the known expressions of Q_0^0, Q_1^0, Q_0^1 by means of the known recurrence relations. *A. Erdélyi* (Pasadena, Calif.)

6080:

★Böhm, C. Sur le cercle de rayon minimum ayant distance nulle d'un ensemble de cercles coplanaires. Symposium on questions of numerical analysis: Proceedings of the Rome Symposium (30 June–1 July 1958) organized by the Provisional International Computation Centre, pp. 33–43. Libreria Eredi Virgilio Veschi, Rome. vii + 79 pp.

L'auteur fournit, pour le problème indiqué dans le titre, une solution en termes finis, puis une solution itérative qui revient à considérer ce problème comme un problème de programmation non linéaire.

J. Kuntzmann (Grenoble)

6081:

Mišković, V. V. Approximations successives dans la méthode des hauteurs égales. Glas Srpske Akad. Nauka **232** Od. Prirod.-Mat. Nauka (N.S.) **15** (1958), 21–27. (Serbo-Croatian. French summary)

6082:

Angelitch, T. P. Über die Verwandlung von quadratischen Formen mit numerischen Koeffizienten in Summen von Quadraten. Z. Angew. Math. Mech. **39** (1959), 160–163.

Mit Hilfe des bei Zurmühl [*Matrizen*, 2. Aufl., Springer, Berlin-Göttingen-Heidelberg, 1958; MR **20** #4567] beschriebenen Schemas kann man eine $n\times n$–Matrix $\mathfrak{A}$ als Produkt von Dreieckmatrizen $\mathfrak{A}+\mathfrak{C}\mathfrak{B}=0$ darstellen. Die Elemente der Matrizen $\mathfrak{B}$ und $\mathfrak{C}$ geben auch Auskunft über Eigenschaften der zugehörigen quadratischen Form $Q=\mathfrak{x}'\mathfrak{A}\mathfrak{x}$. Insbesondere wird Q durch $\mathfrak{y}=-\mathfrak{C}'\mathfrak{x}$ auf die Diagonalform $\bar{Q}=\sum_{i=1}^n b_{ii}y^iy^i$ transformiert, falls Q mindestens ein rein quadratisches Glied enthält. Zwei numerische Beispiele. *J. Schröder* (Hamburg)

6083:

Stenker, Horst; und Sieber, Norbert. Ein Reduktionssatz über Umkehrmatrizen und seine Anwendung auf ein Beispiel aus der Statik. Wiss. Z. Hochsch. Architek. Bauwesen Weimar **6** (1958/59), 105–117. (Russian, English and French summaries)

Es sei A eine nicht-singuläre $n\times n$-Matrix und $\tilde{A}$ eine nicht-singuläre $(n-1)\times(n-1)$-Matrix, welche aus A durch Streichen einer Zeile und einer Spalte entsteht. Dann läßt sich $\tilde{A}^{-1}$ aus A^{-1} in einfacher Weise berechnen. Anwendung auf Probleme der Statik. *J. Schröder* (Hamburg)

6084:

Wilson, L. B. Solution of certain large sets of equations on Pegasus using matrix methods. Comput. J. **2** (1959), 130–133.

This paper discusses one direct and one iterative method for solving linear algebric equations whose coefficients can be represented in the form of a continuant matrix whose elements are themselves matrices. The methods are analogous respectively to Gaussian elimination and Gauss-Seidel iteration, and in general the elimination method is faster, more certain to work, and applicable for matrices of larger order. Comments are given on the Matrix Interpretive Scheme, the existence of which, for the Ferranti Pegasus, facilitates considerably the coding of the methods. *L. Fox* (Oxford)

6085:

Varga, Richard S. *p*-cyclic matrices: A generalization of the Young-Frankel successive overrelaxation scheme. Pacific J. Math. **9** (1959), 617–628.

Der Verfasser untersucht das Iterationsverfahren der Überrelaxation zur Lösung linearer Gleichungssysteme (successive overrelaxation) für den Fall p-zyklischer Matrizen und erhält ähnliche Ergebnisse wie D. Young [Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562] im Fall $p=2$. Das Differenzenverfahren für die ebene Potentialgleichung führt bei einem Dreieckgitter auf 3-zyklische Matrizen. Als weiteres Beispiel wird ein Iterationsverfahren von Peaceman-Rachford [J. Soc. Indust. Appl. Math. **3** (1955), 28–41; MR **17**, 196] behandelt.

J. Schröder (Hamburg)

6086:

Schechter, Samuel. Relaxation methods for linear equations. Comm. Pure Appl. Math. **12** (1959), 313–335.

The convergence of Southwell's relaxation process was established by Temple [Proc. Roy. Soc. London. Ser. A **169** (1939), 476–500] and extended by Ostrowski [Comment. Math. Helv. **30** (1956), 175–210; MR **17**, 898] to allow for under-or-over-relaxation, with a choice of three norms at each step. The present paper extends this investigation to group (or block) relaxation and to residually ordered processes. These techniques seem well adapted for coding for an automatic computer.

G. Temple (Oxford)

6087:

Dorodnicyn, A. A. A contribution to the problem of computing eigenvalues and eigenvectors of matrices. Dokl. Akad. Nauk SSSR **126** (1959), 1170–1171. (Russian)

For symmetric matrices A and B, if the characteristic values and vectors of A are known, the method of perturbation yields those of $A+\varepsilon B$ only if ε is sufficiently small. Nevertheless, no singularities occur, and the values $\lambda_i(\varepsilon)$ and vectors $x_i(\varepsilon)$ satisfy a system of differential equations

$$d\lambda_i/d\varepsilon = x_i^T B x_i, \quad dx_i/d\varepsilon = \sum_{k\neq i} (\lambda_i-\lambda_k)^{-1} x_k^T B x_i\, x_k.$$

Formulas are given resulting from the application of the Euler method, and of the improved Euler method, to the solution of these equations.

A. S. Householder (Oak Ridge, Tenn.)

6088:

Kulik, Stephen. On the solution of algebraic equations. Proc. Amer. Math. Soc. **10** (1959), 185–192.

The method employed is the following to approximate to the zeros of a polynomial $f(x)=(x-a_1)(x-a_2)\cdots(x-a_n)$,

$a_i \neq a_j$ for $i \neq j$ and all a_i real. Differentiate n times the identities $1/f(x) = \sum A_i/(x-a_i)$, $f'(x)/f(x) = \sum 1/(x-a_i)$, thus getting respectively the forms

$$P_{n-1}(x)/[f(x)]^n = \sum A_i/(x-a_i)^n,$$
$$D_{n-1}(x)/[f(x)]^n = \sum (x-a_i)^{-n}.$$

Let X be a number closer to one zero a than to the other zeros a_i. Then as $n \to \infty$

$$X-a = \lim[f(X)P_{n-1}(X)/P_n(X)] = \lim[f(X)^n/P_n(X)]^{1/n},$$
$$X-a = \lim[f(X)D_{n-1}(X)/D_n(X)] = \lim[f(X)^n/D_n(X)]^{1/n}.$$

These equations lead to approximations to the zeros, such as that, if $a_i \leq X$ all i, $\max a_i < X - f(X)P_{n-1}(X)/P_n(X)$. Further approximations use the formulas obtained by differentiating n times the expression $[(u-x)^n f'(x)/f(x)]$, where u is an arbitrary real number.

M. Marden (Milwaukee, Wis.)

6089:

Munro, W. D. Some iterative methods for determining zeros of functions of a complex variable. Pacific J. Math. **9** (1959), 555–566.

Es werden Iterationsverfahren zur Bestimmung der Nullstellen einer komplexen Funktion $f(z)$ hergeleitet, welche im wesentlichen Kombinationen der Bernoullischen Methode und Verschiebungen der Entwicklungsstelle (Horner-Schema) sind. Darunter sind viele bekannte Verfahren, z.B. das Newtonsche. Es sei $g(z)$ eine geeignet gewählte Funktion und $g(z)/f(z) = \sum_{n=0}^{\infty} c_n z^n$. Dann konvergiert die Folge c_n/c_{n+1} unter bestimmten Voraussetzungen gegen eine Nullstelle α kleinsten Betrages von $f(z)$ (Bernoullische Methode). Durch fortwährende Verschiebung der Entwicklungsstelle kann man auch andere Nullstellen berechnen. Der Verf. behandelt die Verfahren $z_{n+1} = z_n + c_N(z_n)/c_{N+1}(z_n)$ mit

$$g(w+z)/f(w+z) = \sum_{n=0}^{\infty} c_n(z)w^n$$

und $z_{n+1} = z_n(1 + c_N(z_n)/C_{N+1}(z_n))$ mit

$$g[z(w+1)]/f[z(w+1)] = \sum_{n=0}^{\infty} c_n(z)w^n.$$

Ist α ein Pol der Ordnung $p=1$ $(p>1)$ des Quotienten $g(z)/f(z)$, so haben diese Verfahren die Ordnung $N+2$ (bzw. 1). Benutzt man in diesen Verfahren eine in bestimmter Weise von z abhängige Funktion $g(w)$, so erhält man im Falle einer einfachen Nullstelle α sogar Konvergenz der Ordnung $N+3$.

J. Schröder (Hamburg)

6090:

Aparo, Enzo. Un criterio di Routh e sua applicazione al calcolo degli zeri di un polinomio. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **25** (1958), 26–30.

An algorithm is set up as an aid to calculating, by electronic computers, the real or complex zeros of a real polynomial. Use is made of Routh's lemma that an nth degree real polynomial $f(z)$ is a Hurwitz polynomial if all $\alpha_{0j} > 0$ $(j = 0, 1, \cdots, r)$ where α_{j0} and α_{j1} are the coefficients in

$$g_0(z) = (1/2)[f(z)+f(-z)] = \alpha_{00} + \alpha_{01}z + \cdots$$
$$g_1(z) = (1/2z)[f(z)-f(-z)] = \alpha_{10} + \alpha_{11}z + \cdots$$

and α_{ij} satisfy the recursion relations $\alpha_{ij} = 0$ for $i >$ degree $g_i(z)$ and

$$\alpha_{ij} = -\begin{vmatrix} \alpha_{0,j-2} & \alpha_{i+1,j-2} \\ \alpha_{0,j-1} & \alpha_{i+1,j-1} \end{vmatrix}$$

for $j > 2$. Use is made of the function defined as $X(F, h) = 1$ or -1 according as $F(z+h)$ is a Hurwitz polynomial or not.

M. Marden (Milwaukee, Wis.)

6091:

★**Aparo, E. L. A stability criterion and its application to the computation of the zeros of a polynomial.** Symposium on questions of numerical analysis: Proceedings of the Rome Symposium (30 June–1 July 1958) organized by the Provisional International Computation Centre, pp. 44–47. Libreria Eredi Virgilio Veschi, Rome. vii+79 pp.

Substantially identical with #6090.

6092:

Stancu, D. D. Contributions à l'intégration numérique des fonctions de plusieurs variables. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. 8 (1957), 75–101. (Romanian. Russian and French summaries)

A number of approximation formulas for the computation of some definite multiple integrals are given. An expression for the remainder is also found for each formula. In particular, the formula of Cavalieri-Simpson for two variables is obtained.

E. Frank (Chicago, Ill.)

6093:

Stancu, D. D. Sur la formule d'interpolation d'Hermite et quelques applications de celle-ci. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. 8 (1957), 339–355. (Romanian. Russian and French summaries)

G. Zemplen [Arch. Math. Phys. (3) 8 (1905), 214–226] gave an expression for the interpolation polynomial of Hermite [J. Reine Angew. Math. **84** (1878), 70–79]:

$$H_n(x) = H_n(x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_s, \cdots, x_s; f|x),$$

where the x_i are repeated r_i times $(i = 1, 2, \cdots, s)$, which the author states is not exact if $s \geqq 2$ and if the numbers r_i are larger than 2. In the present paper formulas for H_n are given. These are used for the decomposition of rational functions in simple fractions, for the integration of functions, and for other applications.

E. Frank (Chicago, Ill.)

6094:

Stancu, D. D. Une méthode de construction des formules de cubature pour les fonctions de deux variables. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Mat. **9** (1958), 351–369. (Romanian. Russian and French summaries)

In this paper the author extends to two variables the method of construction of numerical integration formulas which he states he introduced in his previous study [Com. Acad. R. P. Romîne 8 (1958), 349–358; MR 21 #1475]. This is done with the aid of the interpolation formula of Lagrange-Hermite, extended to two variables [cf. preceding review].

E. Frank (Chicago, Ill.)

6095:

Wilf, Herbert S. A stability criterion for numerical integration. J. Assoc. Comput. Mach. **6** (1959), 363–365.

Die Quadraturformel

$$y_1 = a_0y_0+a_{-1}y_{-1}+\cdots+a_{-p}y_{-p}+h[b_1y_1'+\cdots+b_{-p}y_{-p}']$$

zur Lösung der Differentialgleichung $y'=f(x, y)$ ist genau dann stabil, wenn eine gewisse, aus den a_i, b_i und $\lambda=-h/L$ gebildete Matrix positiv definit ist. Im Beweis wird angenommen, daß man $1/L=\partial f/\partial y$ im betrachteten Gebiet als konstant ansehen kann. *J. Schröder* (Hamburg)

6096:

Bertram, G. Eine Fehlerabschätzung für gewisse selbstadjungierte, gewöhnliche Randwertaufgaben. Numer. Math. **1** (1959), 181–185.

Consider the self-adjoint boundary value problem

$$L[y] \equiv \sum_{\mu=0}^{m} (-1)^{\mu}(p_\mu(x)y^{(\mu)})^{(\mu)} = r(x),$$

$$y^{(\mu)}(a) = y^{(\mu)}(b) = 0 \quad (\mu = 0, 1, \cdots, m-1)$$

with real, non-negative coefficients $p_\mu \in C^\mu[a, b]$ and $p_m(x)\geqq p>0$. Assume that $u(x)$ is the (unique) solution of the problem and that $v(x)\in C^{2m}$ satisfies the boundary conditions and is an approximate solution of the differential equation. The maximal error $|f|_{\max}=\max_{a\leqq x\leqq b}|v(x)-u(x)|$ can be estimated in the form

$$|f|_{\max} \leqq [(b-a)^{2m-1}/2^{2m}pN]\int_a^b |L[v]-r|dx$$

where

$$N = \inf\int_0^1 [s^{(m)}(t)]^2dt.$$

The infimum has to be taken over all $s(t)\in C^{2m}[0, 1]$ with $s^{(\mu)}(0)=0$ $(\mu=0, \cdots, m-1)$, $s^{(m)}(1)=1$. This variational problem has a strong minimal which can be derived explicitly and gives $N=(2m-1)[(m-1)!]^2$. As a conclusion of this interesting paper the estimate obtained for $|f|_{\max}$ is applied to a numerical example.

W. C. Rheinboldt (Syracuse, N.Y.)

6097:

Weinberger, H. F. Lower bounds for higher eigenvalues by finite difference methods. Pacific J. Math. **8** (1958), 339–368; erratum, 941.

In this very remarkable paper, the author first extends to higher eigenvalues, to higher dimensions, to variable coefficients, and to systems of equations, the lower bounds for membrane eigenvalues given by him earlier [Comm. Pure Appl. Math. **9** (1956), 613–623; MR **18**, 826]. For the membrane eigenvalue problem

$$(1) \qquad \Delta u_i+\lambda_i u_i = 0 \text{ in } R, \quad u_i = 0 \text{ on } \dot R,$$

with $\lambda_1\leqq\lambda_2\leqq\cdots$, the author follows his prior work in defining $\lambda_1^{(h)}\leqq\lambda_2^{(h)}\leqq\cdots$ to be the eigenvalues of the usual 5-point difference approximation to Δ over a region R_h containing R and all its leftward and downward translates of distances up to h. The Poincaré inequality [Amer. J. Math. **12** (1890), 211–294] shows that $\lambda_k^{(h)}\leqq$ the maximum of the Rayleigh quotient over any k-dimensional test space. The author takes a test space spanned by the average over mesh squares of the ith eigenfunction u_i $(i=1, \cdots, k)$. He then proves that

$$(2) \qquad \lambda_k \geqq \lambda_k^{(h)}[1+\pi^{-2}h^2\lambda_k^{(h)}]^{-1},$$

for $k=1, 2, \cdots$.

A result of the same type is given for the problem

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x^i}\left(p^i\frac{\partial u}{\partial x^i}\right)+qu = \lambda ru \text{ in } R, \quad u = 0 \text{ on } \dot R,$$

in N dimensions, with $p^i>0$, $r>0$ and $q\geqq 0$ piecewise C^1 in $R\cup\dot R$. For the general self-adjoint problem, it is possible only to prove an implicit lower bound of the order $\lambda_k\geqq\lambda_k^{(h)}-O(h^{1/2})$, as $h\to 0$. However, a bound of the order $\lambda_k\geqq\lambda_k^{(h)}-O(h^2)$ is proved for self-adjoint systems with no mixed derivatives.

Later, the author shows how to give a priori estimates of the difference between upper and lower bounds for λ^k for problem (1), both obtained from difference equations. Here the upper bounds are those of Pólya [C. R. Acad. Sci. Paris **235** (1952), 995–997; MR **14**, 656]. If the boundary of R_h approaches $\dot R$, as $h\to 0$, then both the upper and lower bounds for λ_k converge to λ_k, and explicit estimates are given.

The author next applies his method to the non-homogeneous problem

$$(3) \qquad -\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial u}{\partial x^j}\right)+qu = G \text{ in } R, \quad u = 0 \text{ on } \dot R,$$

where $[a_{ij}]$ is a uniformly positive definite symmetric matrix, $q\geqq 0$, and the coefficients a_{ij}, q are piecewise differentiable, with G continuous. He is able to get a lower bound for the quotient which is minimized by the solution of (3). This can be combined with an upper bound from the Dirichlet principle to get a pointwise approximation to u at interior points of R, following Diaz and Greenberg [J. Math. Phys. **27** (1948), 193–201; MR **10**, 213] and Greenberg [ibid., 161–182; MR **10**, 117].

A final section treats the vibrating clamped plate as an example of an extension to certain elliptic operators of higher order. *G. E. Forsythe* (Stanford, Calif.)

6098:

Davidenko, D. F. Solution by the method of nets of the axi-symmetric Dirichlet problem for the Laplace equation. Dokl. Akad. Nauk SSSR **126** (1959), 471–473. (Russian)

In a previous paper [same Dokl. **110** (1956), 110–113; MR **18**, 827] the author develops a method of solving the Dirichlet problem for $r^{-1}u_r+u_{rr}+u_{zz}=0$ on a rectangular grid using a 9-point formula giving remainders of order h^8 off the axis and h^6 on. Here the method is refined and simplified slightly.

A. S. Householder (Oak Ridge, Tenn.)

6099:

Davidenko, D. F. Numerical determination of a Stokes flow function. Dokl. Akad. Nauk SSSR **126** (1959), 699–702. (Russian)

The method previously developed [reference in review above] for the solution of the Dirichlet problem is here applied to the equation $u_{zz}-r^{-1}u_r+u_{rr}=0$.

A. S. Householder (Oak Ridge, Tenn.)

6100:

★**Pucci, Carlo. On the improperly posed Cauchy problems for parabolic equations.** Symposium on the numerical treatment of partial differential equations with real characteristics: Proceedings of the Rome Symposium (28–29–30 January 1959) organized by the Provisional International Computation Centre, pp. 140–144. Libreria Eredi Virgilio Veschi, Rome, 1959. xii + 158 pp.

This note contains the statement (without proof) of a theorem concerning the approximate Cauchy problem

$$|u(0, t)-\varphi(t)| \leq \varepsilon, \quad |u_x(0, t)-\psi(t)| \leq \varepsilon,$$

where $\varphi(t)$ and $\psi(t)$ are continuous, for the equation of heat $u_t = u_{xx}$ in the region R, $0<x<1$, $0<t<T$.

Let B be a closed set contained in the union of R and the t-axis. Let $\delta_\varepsilon(B) = \sup|u(x, t)-u'(x, t)|$ where u, u' are any two positive solutions of the approximate Cauchy problem and (x, t) belongs to B. Then there exists a non-decreasing function $\omega(\varepsilon, M)$ such that $\delta_\varepsilon(B) \leq \omega(\varepsilon, M)$ if $|\phi| \leq M$, $|\psi| \leq M$ and such that $\omega(\varepsilon, M) \to 0$ as $\varepsilon \to 0$. This implies that the positive solutions of the ordinary Cauchy problem for the equation of heat depend continuously on the Cauchy data on the t-axis.

E. T. Copson (St. Andrews)

6101:

Bellman, Richard; and Dreyfus, Stuart. Functional approximations and dynamic programming. Math. Tables Aids Comput. **13** (1959), 247–251.

Remarks concerning the application of dynamic programming to n-dimensional problems in connection with present day computing devices. For $n \geq 3$ the storage of the values of a function f at the $(1/d)^n$ vertices of a grid of size d is prohibitive. The authors advocate the approximation of f by means of linear combinations of products of normalized Legendre (or other) polynomials of a single variable. Integrations should then be performed by interpolation formulas with Christoffel coefficients.

L. Cesari (Lafayette, Ind.)

6102:

Łukaszewicz, L. Accumulation of errors in approximate calculations. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 233–239. (Russian summary, unbound insert)

From data $y_1 = x_1, \cdots, y_P = x_P$, a "numerical sequence" defines $y_{P+1} = w_{P+1}, \cdots, y_N = w_N$, where the w's represent arithmetic operations performed upon y's of lower index. From appropriate hypotheses, the author derives formulas for the mean and variance of y_N and illustrates their application, while calling attention to possible snares and pitfalls.

A. S. Householder (Oak Ridge, Tenn.)

6103:

Ballhausen, C. J.; and Ancmon, E. M. Table of Ligand field integrals. Mat.-Fys. Medd. Danske Vid. Selsk. **31** (1958), no. 9, 1–38.

"Numerical values of integrals occurring in ligand field calculations are tabulated as functions of the 'effective' charges of the wave functions and of the bond lengths. The integrals fall into three classes; 1: The electronic interaction integral $G_{a,b}{}^n = \int R(a)r_<{}^n/r_>{}^{n+1}R(b)r^2dr$ with (α) $a=b=3d$, $n=0$, 2 and 4, (β) $a=3d$, $b=4p$, $n=1$ and 3, (γ) $a=3d$, $b=4s$, $n=2$ and (δ) $a=b=4s$, $n=0$. 2: The first derivatives $B_{a,b}{}^n = dG_{a,b}{}^n/dr_0$ with respect to the bond length r_0. Tabulated are (α) $a=b=3d$, $n=0$, 2 and 4, and (β) $a=3d$, $b=4p$, $n=1$ and 3. 3: The second derivatives $C_{a,b}{}^n = d^2G_{a,b}{}^n/dr_0{}^2$. Tabulated are ($\alpha$) $a=b=3d$, $n=0$, 2 and 4. All the integrals are evaluated using hydrogenlike wavefunctions." (Authors' summary)

J. C. P. Miller (Cambridge, England)

COMPUTING MACHINES

See also 5549.

6104:

Gauss, E. J. A comparison of machine organizations by their performance of the iterative solution of linear equations. J. Assoc. Comput. Mach. **6** (1959), 476–485.

6105:

Moisil, Gr. C. Sur un théorème d'existence dans la théorie algébrique des mécanismes automatiques discrets. Rev. Math. Pures Appl. **3** (1958), 9–23.

This paper is concerned with an algebra of discrete automatic mechanisms, that is, information processing machines. An ideal machine is postulated which has a finite number of input states (describable as buttons which can be on or off), contacts corresponding to the inputs, currents through the contacts which may or may not flow, and output states (pictured as lights which may be lighted or not). Given a sequence of inputs the various states of the machine can be described by a set of Boolean equations, called an "exact program". An exact program could have internal contradictions, but if free of contradictions is called "compatible". If there is a machine which performs an exact program then the program is "realizable".

The theorem is that an exact program is realizable if and only if it is compatible. *H. H. Campaigne* (Jessup, Md.)

6106:

Rosenblatt, Frank. Perceptron simulation experiments. Proc. I.R.E. **48** (1960), 301–309.

An experimental simulation program, which has been in progress at the Cornell Aeronautical Laboratory since 1957, is described. This program uses the IBM 704 computer to simulate perceptual learning, recognition, and spontaneous classification of visual stimuli in the perceptron, a theoretical brain model which has been described elsewhere. The paper includes a brief review of the organization of simple perceptrons, and theoretically predicted performance curves are compared with those obtained from the simulation programs, in several types of experiments, designed to study "forced" and "spontaneous" learning of pattern discriminations.

Author's summary

6107:

Hagensick, Paul W. Logic by machine: programming the LGP-30 to solve problems in symbolic logic. Behavioral Sci. **5** (1960), 87–94.

6108:

Green, Bert F., Jr. IPL-V: The Newell-Shaw-Simon programming language. Behavioral Sci. **5** (1960), 94–98.

6109:

Nagler, H. Amphisbaenic sorting. J. Assoc. Comput. Mach. **6** (1959), 459–468.

The method for sorting tape files is derived from the method usually known as distribution sorting. If there are $p+1$ tape units available the keys are expressed to radix p (convenient algorithms are given for this) and the digits of the keys are examined in turn starting with the most significant. At the first reading the file is distributed into p groups designated $0, 1, \cdots, p-1$ and written onto tape units $t_0, t_1, \cdots, t_{p-1}$. At every other pass the group with the highest numerical value is distributed among the other p tape units; thus the $p-1$ group is first subdivided into $(p-1, 0)$, $(p-1, 1)$, $\cdots$, $(p-1, p-1)$ subgroups and these are written onto $t_0, t_1, \cdots, t_{p-2}, t_p$. Eventually a group is obtained with all keys identical, whereupon it is written onto t_p, which contains the sorted file, or the group becomes small enough so that it is advantageous to sort it internally, after which it is written onto t_p. By keeping track of the number of records in each group it is known whether internal sorting is possible and from the range of keys it can be decided whether radix or merge sorting is preferable. *C. C. Gotlieb* (Toronto, Ont.)

6110:

★**McCracken, Daniel D.; Weiss, Harold; and Lee, Tsai-Hwa. Programming business computers.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1959. xvii+510 pp. $10.25.

This book is, in its authors' words, "written for the person who is involved (or expects to be) in the day-to-day application of electronic computers to business data processing problems, or whose work is so closely related to computer applications that he must have something more than a skimming of the highlights". For the people to whom it is addressed, the book should prove eminently satisfactory. Mathematicians might find it somewhat tedious and wordy, and "top management" people interested only in the "broad picture" would find it somewhat over their heads.

Like the only other book addressed to a comparable audience, Gotlieb and Hume's *High-speed data processing* [McGraw-Hill, New York, 1958; MR **20** #2864], this book is based on a hypothetical machine which the authors call DATAC. Since DATAC has an instruction repertoire and other properties characteristic of a broad class of actual machines, and since significant quirks present in this or that actual machine but absent in DATAC are carefully signalled when the occasion warrants, the reader should have little difficulty in applying in practice what he has learned about DATAC.

The book is very clearly written, and the typographical layout is attractive and easy on the eyes. The reader is expected to have little mathematical background, if any; hence such mysteries as decimal point placement in multiplication and division are unveiled by examples. In contrast to Gotlieb and Hume, the authors make no attempt to present details of the logical design of such components as adders, and such questions as binary-to-decimal conversion are relegated to an appendix.

Significant topics, ranging from the organization of files, through flow charting, the use of index registers, and so on, to an introduction to machine-aided coding (which the authors wisely refrain from calling automatic programming) are well presented in sufficient detail and with many well-chosen examples. Each chapter has a useful summary and a good set of exercises. The book abounds in homey admonitions to avoid pitfalls such as, for example, "It is a common delusion of newcomers to the data processing field to assume that an application, once programmed, is finished. The ground rules in practice change continuously, so the wise programmer makes allowances for easy maintenance". Such comments, and their tone, may exasperate the experienced, and indeed the authors occasionally apologize for making them, but experience shows them to be necessary, and one hopes that the readers to whom the book is addressed will take them to heart. In what might be charitably called an excess of enthusiasm, manufacturers often avoid discussing pitfalls, and it is important to have them recorded and brought to the attention of those who will encounter them.
A. G. Oettinger (Cambridge, Mass.)

6111:

Lorente, Gabriel. Remarques sur une théorie base de l'apprentissage. Cybernetica **2** (1959), 127–135.

Comparison entre l'homme et la machine, sans indication d'applications précises. *J. Kuntzmann* (Grenoble)

MECHANICS OF PARTICLES AND SYSTEMS

See also 5936, 6021, 6249, 6254.

6112:

Volmer, J. Les cas particuliers de la courbe des centres de Burmester, à point double, et leur importance pour la théorie des mécanismes. Acad. R. P. Romîne. Stud. Cerc. Mec. Apl. **10** (1959), 425–448. (Romanian. Russian and French summaries)

Quand on considère quatre positions d'un plan mobile par rapport à un plan fixe V, il y a des quadruples de points homologues qui se trouvent sur une circonférence de cercle; le lieu des centres est la courbe C dans V mentionnée dans le titre ("Mittelpunktskurve" de Burmester). On peut faire emploi de la courbe C dans certaines constructions qui sont importantes pour la synthèse des mécanismes. L'auteur étudie le cas où les quatres positions sont deux à deux infiniment voisines. Alors les centres de rotation P_{12} et P_{34} sont généralement distincts, mais les autres P_{ij} se confondent dans un point P. La courbe C est du troisième degré, passant par P_{12} et P_{34} et ayant point double à P. Plusieurs cas particuliers sont distingués (par exemple: le triangle $PP_{12}P_{34}$ est équilatéral, le point P se trouve à l'infini, etc.). *O. Bottema* (Delft)

6113:

Bodner, V. A.; Ovcharov, V. E.; and Seleznev, V. P. On the synthesis of invariant damped inertial systems of arbitrary period. Dokl. Akad. Nauk SSSR **125** (1959), 986–988 (Russian); translated as Soviet Physics. Dokl. **4**, 303–305.

6114:

Miele, Angelo. Lagrange multipliers and quasi-steady flight mechanics. J. Aero/Space Sci. **26** (1959), 592–598.

"A general method is presented for investigating optimum conditions of the quasi-steady mechanics of flight.

"For motion in a vertical plane the problem of extremizing an arbitrarily specified function of altitude, Mach number, lift, path inclination, engine control parameter, and thrust inclination is considered for an aircraft which has to satisfy the equations of motion and three additional arbitrary constraints.

A generalized solution is obtained in a determinantal form. The characteristic of this solution is that it unifies into a single equation the results of a large segment of the previous contributions to the quasi-steady mechanics of flight. Particular problems such as maximum speed, maximum range, maximum endurance, ceiling, steepest ascent, best rate of climb, flattest descent, etc., are thus all covered by the same determinantal equation." (From the author's summary.) *P. O. Bell* (Culver City, Calif.)

6115:

Garber, T. B. On the rotational motion of a body re-entering the atmosphere. J. Aero/Space Sci. **26** (1959), 443–449.

6116:

Harlamova-Zabelina, E. I. Rapid rotation of a solid body about a fixed point with non-holonomic constraints. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 6, 25–34. (Russian)

6117:

Goodstein, Robert. A perturbation solution of the equations of motion of a gyroscope. J. Appl. Mech. **26** (1959), 349–352.

The author applies a perturbation method to the nonlinear differential equations of motion of free and forced vibration of a two-gimbal gyro. He gives special consideration to the problem in which the gyro housing is subjected to a sinusoidal angular displacement such as from the motion of a missile or other carrier vibrating as an elastic body. He gives some practical interpretation to his mathematical results. *H. P. Thielman* (Ames, Iowa)

6118:

Novoselov, V. S. On the motion of gyroscopic systems. J. Appl. Math. Mech. **23** (1959), 242–246 (176–178 Prikl. Mat. Meh.).

The author considers the case when the differential equations of a gyroscopic system have variable coefficients. His suggested method for solving the system of equations is based on a substitution of new variables for the independent variables. *H. P. Thielman* (Ames, Iowa)

6119:

Leitmann, G. On a class of variational problems in rocket flight. J. Aero/Space Sci. **26** (1959), 586–591.

"The problem considered is that of determining the direction and magnitude of the bounded thrust of a single-stage rocket, traveling in vacuum and a constant gravitational field, in order to minimize a function of the initial and final values of rocket mass, position and velocity components, and time. The general nature of the extremal arc is derived from a consideration of necessary conditions for the existence of a minimum 'pay-off'. Criteria are established for the identification of extremal subarcs and for their composition into the complete extremal arc. It is found that flight takes place either at maximum or minimum thrust and that, at most, three such subarcs can arise." (Author's summary)

P. O. Bell (Culver City, Calif.)

STATISTICAL THERMODYNAMICS AND MECHANICS

See also 6132.

6120:

Antončík, Emil. The use of the repulsive potential in the quantum theory of solids. Czechoslovak J. Phys. **9** (1959), 291–305. (Russian summary)

The calculation of the energy levels of electrons in crystals is complicated greatly by the necessity of orthogonalization of the wave functions. One device which has been suggested in the literature [P. Gombás, Handbuch der Physik, vol. 36, pp. 139–164, Springer, Berlin, 1956] for replacing the orthogonalization process is that of making the hamiltonian operator dependent on the energy level to be calculated. This is accomplished by introducing an effective repulsive potential which depends on the wave functions of the states of lower energy. The variational principle can then be employed to estimate the lowest eigenvalue of this altered hamiltonian operator, by relatively simple calculations. The author studies the effectiveness of this procedure in the case of silicon, and shows that it compares favorably with other methods of calculation. *E. L. Hill* (Minneapolis, Minn.)

6121:

★**Sandmeier, Armin Henry. The kinetics and stability of fast reactors with special considerations of nonlinearities.** Thesis, Swiss Federal Institute of Technology Zürich. Juris-Verlag, Zürich, 1959. 89 pp.

This thesis discusses the kinetics of fast-neutron reactors (those in which fissions are mainly produced by fast neutrons), and analyzes in detail some of the specific non-linearities introduced into the kinetic behavior of such reactors by changes in density within the reactor, changes in cross sections due to the Doppler effect, and changes in geometrical shape due to heating.

R. R. Coveyou (Oak Ridge, Tenn.)

6122:

Eriksson, Karl-Erik. Statistical time moments and an asymptotic formula for the time-energy distribution of slowed-down neutrons. Ark. Fys. **16** (1959), 1–14.

Calculations relating to the time-energy distribution of neutrons slowing down in a homogeneous monatomic moderator, including a derivation of an asymptotic

formula for the time-energy distribution. The applicability of the results is limited by a quite special choice of dependence of mean free path on energy.
R. R. Coveyou (Oak Ridge, Tenn.)

6123:
★Kubo, Ryogo. Some aspects of the statistical-mechanical theory of irreversible processes. Lectures in theoretical physics, Vol. I. Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1958 (edited by W. E. Brittin and L. G. Dunham), pp. 120–203. Interscience Publishers, New York-London, 1959. vii+414 pp. $6.00.

A discussion of the foundations of irreversible thermodynamics. First the Onsager relations are derived, and then their implications for the dynamics of linear dissipative systems are explored. A number of specific problems are treated, including electrical conductivity in a magnetic field, and the motional narrowing of spectral lines.
H. W. Lewis (Madison, Wis.)

6124:
Leech, J. W. Irreversible thermodynamics and kinetic theory in the derivation of thermoelectric relations. Canad. J. Phys. **37** (1959), 1044–1054.

A discussion is given of the thermoelectric coefficients, first on the basis of irreversible thermodynamics, and then using the transport equation. *D. ter Haar* (Oxford)

ELASTICITY, PLASTICITY

See also 6154, 6156.

6125:
Goded, F. The stress function of a radial strain. J. Appl. Mech. **26** (1959), 440–441.

"Author derives a stress function to study radial strain and, in particular, that in a cone." (From the author's summary) *H. D. Conway* (Ithaca, N.Y.)

6126:
Solov'ev, Iu. I. The action of a concentrated force on an eccentric ring. J. Appl. Math. Mech. **22** (1958), 989–996 (701–705 Prikl. Mat. Meh.).

The stress function and stress components which result from the application of a concentrated force to either periphery of the region bounded by two eccentric circles are found by using bipolar coordinates and expanding the stress function as a Fourier series, the coefficients of which are determined from the boundary conditions. As examples the cases of an eccentric ring acted on by two equal and opposite forces, applied along a diameter of the outer circle, and a half plane containing a circular hole acted on by a force applied at a point of the hole, are treated numerically.
W. D. Collins (Newcastle-upon-Tyne)

6127:
Satō, K. Large deflection of a circular cantilever beam with uniformly distributed load. Ing.-Arch. **27** (1959), 195–200.

Using the Bernoulli-Euler equation, the author solves the problem of a thin circular cantilever beam, convex downward, under a uniform load. The investigation is divided into two parts: (a) with the deflections sufficiently small for the initial and final beam curvatures to have the same sign; (b) with one inflection point in the elastic line. A power series method of solution is adopted, several numerical cases being worked out.
H. D. Conway (Ithaca, N.Y.)

6128:
Woodhead, R. W. An approximate partial differential equation for the bending of an unsymmetrical composite plate. Austral. J. Appl. Sci. **10** (1959), 123–137.

The bending equilibrium equations and associated boundary conditions are derived by variational methods for the small elastic deflection of a particular composite plate. The plate consists of three layers, each of which is homogeneous and isotropic. The first is a classical plate with transverse shear deformation considered, the second is capable of transmitting only shear and normal force, and the third is a simple membrane. The stress distribution assumption violates some of the boundary conditions and hence leads to approximate solutions. Some examples of practical application of the theory are indicated.
S. R. Bodner (Providence, R.I.)

6129:
Fraeijs de Veubeke, B. Flexion et extension des plaques d'épaisseur modérée. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in-8° (2) **31** (1959), no. 6, 38 pp.

6130:
Gerisch, Wolfgang. Zum Problem der an den Rändern fest eingespannten, elastischen Platte. Arch. Math. **10** (1959), 298–303.

The operator

$$A=\frac{\partial^4}{\partial x^4}+2\alpha^2\frac{\partial^4}{\partial x^2\partial y^2}+\alpha^4\frac{\partial^4}{\partial y^4}+\theta\alpha\frac{\partial^2}{\partial x\partial y}$$

with $\alpha\geqq 1$, $\theta\geqq 0$, occurs in the theory of elastic plates under shear. For the region S: $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, and "clamped" boundary conditions $f=\partial f/\partial n=0$, A is shown to be bounded below:

$$\iint A\,f(x,y)\overline{f(x,y)}dS\geq -\theta^2/16\iint|f(x,y)|^2dS.$$

R. C. T. Smith (Armidale)

6131:
★Novozhilov, V. V. The theory of thin shells. Translated by P. G. Lowe. Edited by J. R. M. Radok. P. Noordhoff Ltd., Groningen, 1959. xvi+376 pp. Paperbound: $9.50; Dfl. 36.00.

English translation of *Teoriya tonkih oboloček* [Gosudarstv. Izdat. Sudostroit. Lit., Moscow, 1951; MR **17**, 915]. It includes a preface by the author, written for this translation.

6132:
Seeger, Alfred; und Mann, Erich. Anwendung der nichtlinearen Elastizitätstheorie auf Fehlstellen in Kristallen. Z. Naturf. **14a** (1959), 154–164.

The theory of elasticity is a helpful tool in the investigation of defects of crystals. However, some phenomena are outside the range of the linear theory. The authors consider an extension of the theory by taking into account the physical nonlinearity in the stress-strain relations; geometric nonlinear terms are still neglected in their analysis. Solution of some problems is effected by a perturbation method. Results are in agreement with Zener's theorem for average dilatation in a homogeneous body in a state of residual stress [Trans. Amer. Inst. Mining Metallurg. Engrs. **147** (1942), 361]. *W. T. Koiter* (Delft)

6133:

Borş, C. I. La fonction de tension et le problème de la torsion des barres formées de plusieurs matériaux anisotropes. An. Şti. Univ. "Al. I. Cuza" Iaşi. Secţ. I. (N.S.) **4** (1958), 81–87. (Romanian and Russian summaries)

This paper deals with the torsion function for a cylindrical bar consisting of several parallel cylindrical rods made of different anisotropic materials but all having a plane of elastic symmetry perpendicular to their lengths. The cross section of the bar is made up of several domains $S_1, S_2, \cdots, S_m$ corresponding to the component rods and a domain S_0 corresponding to the surrounding material. The boundary curves L_i of each domain S_i $(i=1, 2, \cdots, m)$ are assumed to be simple closed curves and the boundary curve of S_0 will consist of the closed curves L_i $(i=1, 2, \cdots, m)$ together with a curve L_0 which encloses all the others. As special cases of the general result a composite bar for which L_i $(i=1)$ and L_0 are (i) two concentric circles and (ii) two confocal ellipses, are dealt with.

R. M. Morris (Cardiff)

6134:

Laasonen, Pentti. Eigenoscillations of an elastic cable. Quart. Appl. Math. **17** (1959), 147–154.

The system of partial differential equations of vibration of a flexible cable supported at two points of a horizontal plane is written and linearized in terms of perturbations of displacements from the equilibrium position. The horizontal and vertical components of the perturbed displacements in the plane of equilibrium are then assumed to have the form $u(q)e^{i\omega t}$ and $v(q)e^{i\omega t}$, where q is a measure of distance to the point along the cable. The principal eigenvalue problem then takes this form:

$$\frac{d}{dq}\left[\frac{(1+\varepsilon\phi^3)u'+\alpha q v'}{\varepsilon(1+\varepsilon\phi)\phi^2}\right]+\lambda u = 0,$$

$$\frac{d}{dq}\left[\frac{\alpha q u'+(\alpha^2q^2+\varepsilon\phi^3)}{\varepsilon(1+\varepsilon\phi)\phi^2}\right]+\lambda v = 0,$$

$u(\pm 1)=v(\pm 1)=0$, where $\phi(q)=(1+\alpha^2q^2)^{1/2}$, α and ε are positive constants, and the eigenvalue parameter λ is proportional to the frequency ω. The self-adjointness of the problem is examined and an orthogonality property is noted. For shallow cables α and ε are small. The author examines asymptotic eigenvalues for small α and ε, and makes some observations about the lowest eigenvalues in practical cases. *R. V. Churchill* (Ann Arbor, Mich.)

6135:

Beesack, Paul R. Isoperimetric inequalities for the nonhomogeneous clamped rod and plate. J. Math. Mech. 8 (1959), 471–482.

The main purpose of this paper is the estimation by upper and lower bounds of the least eigenvalue for a non-homogeneous vibrating plate of constant flexural rigidity which is clamped along its edge. As an introduction the similar problem in the simpler case of a clamped rod is first investigated.

The principal idea (especially suitable for this kind of problem) is to compare, in the case of the rod, the eigenvalue for the general density distribution with those for a symmetric decreasing or increasing distribution about the midpoint and to compare in the case of the plate the eigenvalue with those of similar circular symmetric functions. To make this comparison precise conveniently, the densities are rearranged in an equimeasurable manner, which means that the two sets of points of the symmetrical and the non-symmetrical distribution, where the density is greater than an arbitrary constant, must have the same Lebesgue measure. Then the following theorems are true: The eigenvalue for the rearranged symmetrically decreasing [or increasing] density bounds the eigenvalue of the general distribution from below [or above]. The proofs are based upon: (1) the basic theorem of rearrangement of functions given by Hardy, Littlewood and Polya [*Isoperimetric inequalities*, 2nd ed., Univ. Press, Cambridge, 1952; MR **13**, 727]; (2) the minimum property of the least eigenvalue; (3) a theorem on the non-vanishing of the first eigenfunction [see S. A. Janczewsky, Ann. of Math. (2) **29** (1928), 521–542]; and (4) the isoperimetric inequality. As an important tool the author uses the Cauchy-Schwarz inequality at several points of his proofs.

The reviewer believes that the paper presents a useful contribution to the well-known class of isoperimetric inequalities occurring in mathematical physics. The precision with which the proofs are built up and with which previous proofs on parts of the subject are corrected may suggest further critical work in this field.

W. Schumann (Zürich)

6136:

Dyer, Ira. Response of plates to a decaying and convecting random pressure field. J. Acoust. Soc. Amer. **31** (1959), 922–928.

The correlation of vibration and mean square response of a simply supported plate are computed assuming a two-dimensional random pressure field with an exponential type autocorrelation function. The physical interpretation of the results is discussed.

E. Reich (Minneapolis, Minn.)

6137:

Cross, Arthur K. Generalized spectral representation in aeroelasticity. J. Aero/Space Sci. **26** (1959), 766–767.

A particular form of the spectral theorem for Hermitian matrices is given. It is also explained how this theorem can be used in certain aeroelastic calculations. In the statement of the theorem, the rank of the matrices involved is permitted to be less than the order. However the matrices are assumed to have inverses, so the rank must equal the order. *J. B. Keller* (New York, N.Y.)

6138:

Voitsekhovskaia, K. F. The stability of a cylindrical shell from the standpoint of the mathematical theory of

elasticity. Soviet Physics. Dokl. **123** (3) (1958), 1279–1282 (623–626 Dokl. Akad. Nauk SSSR).

A circular cylindrical shell is compressed in the axial direction. The axially symmetric form of instability is examined on the basis of exact equations of the mathematical theory of elasticity but neglecting the components of rotation. A numerical example shows less than 0.04 per cent difference between the values of critical stress thus calculated and from a thin shell theory.

L. S. D. Morley (Farnborough)

6139:

Darevskii, V. M.; and Kukudzhanov, S. N. Torsional stability of an orthotropic shell under internal pressure. Soviet Physics. Dokl. **123** (3) (1958), 1271–1274 (49–52 Dokl. Akad. Nauk SSSR).

The problem of the title is investigated, the length l of the shell being in the range

$$\epsilon^{1/2} \max(K, K^{-1}) \ll (\pi R/l)^2 \ll \epsilon^{-1/2} \min(K, K^{-1/2}),$$

where $\epsilon = h^2/12R^2(1-\nu_1\nu_2)$, $K=\sqrt{(E_2/E_1)}$; h and R being, respectively, the thickness and radius, and E_1, E_2 and ν_1, ν_2 being, respectively, the elastic moduli and Poisson's ratio in the axial and circumferential directions. The shell edges are assumed to be either simply-supported or clamped.

H. D. Conway (Ithaca, N.Y.)

6140:

Yamaki, Noboru. Postbuckling behavior of rectangular plates with small initial curvature loaded in edge compression. J. Appl. Mech. **26** (1959), 407–414.

A solution is presented for a square plate with small initial deviations from flatness, subjected to edge compression resulting in large deflections. Eight combinations of edge clamping conditions are considered. The large deflection F-equation is satisfied analytically and the w-equation is approximately satisfied by Galerkin's method. Numerical values are presented for a variety of loadings. Results show that initial lack of flatness has marked effects on the effective width for loads near the buckling load.

S. Levy (Philadelphia, Pa.)

6141:

Haywood, J. H. Response of an elastic cylindrical shell to a pressure pulse. Quart. J. Mech. Appl. Math. **11** (1958), 129–141.

This paper gives an approximate solution to the elastic response of an infinitely long uniform circular cylindrical shell which is in an acoustic medium and is subjected to a lateral plane pressure pulse. A modal method of analysis is employed in which dilatational, translational and inextensional-flexural modes are considered. An approximation to a solution of the cylindrical wave equation leads to a relation between the fluid pressure and the particle velocity which is incorporated into the equations of motion. Very good agreement is obtained with some exact solutions.

S. R. Bodner (Providence, R.I.)

6142:

Ambraseys, N. N. On the shear response of a two-dimensional truncated wedge subjected to an arbitrary disturbance. Bull. Seismol. Soc. America **50** (1960), 45–56.

"The present work consists of a theoretical investigation of the shear response of a truncated two-dimensional elastic wedge subject to an arbitrary disturbance. Expressions are derived for the deflections and shears which develop in the wedge owing to an imposed time-dependent disturbance. The frequencies of the wedge are derived for the six first modes of oscillation and are given graphically for different degrees of truncation for the one and two-dimensional cases. The solution derived is applicable to earthquake engineering problems, in particular to those dealing with the seismic stability of earth dams and embankments. The concept of strong ground-motion spectra is introduced and its advantages and limitations are discussed briefly."

Author's summary

6143:

Ben-Menahem, Ari. Diffraction of elastic waves from a surface source in a heterogeneous medium. Bull. Seismol. Soc. America **50** (1960), 15–33.

"A theory is developed for wave propagation of a given frequency emerging from a seismic surface source in a medium in which the velocity is a continuous function of one coordinate only. It is assumed that the relative change of the elastic parameters is very small over a wave length. The wave equations are then solved in cylindrical coordinates under suitable boundary conditions and integral representations are obtained for the displacements, which are generally valid. These integrals are then evaluated for a special case with an almost linear velocity gradient and the surface displacements are obtained for long ranges. It is found that the amplitude of the body waves (both P and S) inside the shadow zone decays exponentially with the distance from the source at a rate proportional to one-third power of the frequency and two-thirds power of the velocity gradient."

Author's summary

6144:

Caughey, T. K. Response of a nonlinear string to random loading. J. Appl. Mech. **26** (1959), 341–344.

The usual wave equation for the transverse forced motion of a string subject to damping is slightly modified. The modification consists in adding to the constant tension a term which takes account of the stretching of the string. The solution of the resulting nonlinear equation is sought for a finite string fixed at its endpoints $x=0$, L. Initial conditions are not specified. The solution $u(x, t)$ is approximated by

$$u(x, t) = \sum_{i=1}^{N} b_i(t) \sin \frac{i\pi x}{L}.$$

A coupled system of nonlinear ordinary differential equations is obtained for the $b_i(t)$. These equations are approximated by a set of uncoupled linear equations. The restoring force in each equation, which is chosen to minimize the error, depends upon the mean square values of all the b_i's. Each $b_i(t)$ is found to be a Gaussian process since the forcing term in each equation is assumed to be Gaussian. Finally the mean square deflection of the string is computed and found to be less than that for a linear string. The calculation involves various approximations, most of which are explicitly mentioned.

J. B. Keller (New York, N.Y.)

6145:

Olszak, W.; and Zahorski, S. A non-homogeneous orthotropic circular segment as an elastic-plastic problem. Arch. Mech. Stos. **11** (1959), 409–419. (Polish and Russian summaries)

The authors consider a segment of a circular beam with distinct finite radii a, c, b which is bent by a monotonically increasing terminal couple M. The material is assumed to be axially orthotropic but may be non-homogeneous. Further restrictions are introduced in the course of the analysis. The following results are obtained: (1) The defining equations for an elastic material; (2) explicit expressions for the stresses in an incompressible elastic material; (3) explicit expressions for the stresses in the fully plastic state of a perfectly plastic material which yields according to an anisotropic generalization of the Mises yield condition, together with a formula for the corresponding moment M_3; (4) restrictions on the elastic and plastic coefficients under which plastic flow will begin at $r=a$ under a moment M_1 and a second plastic region will form at $r=b$ under a moment M_2; explicit expressions for M_1 and M_2. *P. G. Hodge, Jr.* (Chicago, Ill.)

6146:

Ivlev, D. D. On the development of a theory of ideal plasticity. J. Appl. Math. Mech. **22** (1958), 1221–1230 (850–855 Prikl. Mat. Meh.).

Much of the paper is devoted to a review of constitutive laws for isotropic rigid, perfectly plastic solids. The following assumptions are made: the mechanical properties are independent of the mean normal stress, the yield locus is symmetric with respect to the origin of stress space, and the theory of the generalized plastic potential is valid. For a given strain rate tensor, Tresca's yield condition and the associated flow rule then predict a power of plastic dissipation that does not exceed the power of dissipation obtained from any other yield condition with the same yield stress in simple tension. The author seems to feel that this fact provides strong theoretical support for Tresca's yield condition, and that any observed deviations from the behavior predicted by this condition should be attributed to secondary effects.

W. Prager (Providence, R.I.)

6147:

Rabotnov, Iu. N. Model illustrating some properties of a hardening plastic body. J. Appl. Math. Mech. **23** (1959), 219–228 (164–169 Prikl. Mat. Meh.).

The author explores possible types of behavior of an elastic, perfectly plastic material under combined states of stress by considering a thin-walled circular tube of such a material that is subjected to the bending moments M_x and M_y in two orthogonal diametral planes. Proportional loading (during which $M_x/M_y=\text{const.}$) corresponds to simple flexure, for which the relations between the bending moments M_x, M_y and the curvatures K_x, K_y are readily established. Through a given stress point M_x, M_y, there exist loading paths deviating from proportional loading, for which these relations remain valid because no element of the tube experiences unloading. Similarly, through a given stress point, there exist paths of purely elastic unloading that deviate from proportional unloading. For all other stress paths, incremental relations can be established between the bending moments and the curvatures. The incremental relations, though more complex than those used in existing theories of plasticity, nevertheless satisfy generally accepted principles of these theories, for example, the principle of maximum power of dissipation. *W. Prager* (Providence, R.I.)

6148:

Cristescu, Nicolae. Sur l'effet de Bauschinger. C. R. Acad. Sci. Paris **249** (1959), 616–618.

The paper is concerned with incompressible work-hardening plastic solids that exhibit Bauschinger effect. In the octahedral plane of stress space, the initial yield locus is assumed to be the Tresca hexagon or the Mises circle. Several assumptions regarding the displacement and deformation of the yield locus in the course of work-hardening are discussed. The author does not seem to be aware of the fact that these and other assumptions have already been widely used in numerous theoretical and experimental papers published in the U.K., the U.S., and the U.S.S.R. during the last decade.

W. Prager (Providence, R.I.)

6149:

Sternberg, Eli; and Chakravorty, J. G. Thermal shock in an elastic body with a spherical cavity. Quart. Appl. Math. **17** (1959), 205–218.

The thermoelastic problem for an infinite isotropic elastic medium with a spherical cavity is investigated, the boundary of the cavity being subjected to a sudden uniform change of temperature. Departing from the conventional quasi-static treatment, the inertia terms are introduced into the governing field equation, thus leading to a solution to an exact elastokinetic problem. In this connection the present paper establishes a new solution in the little explored field of dynamic thermoelasticity, given by the authors after a solution [J. Appl. Mech. **26** (1958), 503–509] which completed and extended the results by Danilovskaya.

The solution of the heat-conduction problem under consideration being known, the governing radial displacement (u) field equation

$$(*)\qquad u_{,\rho\rho}+2/\rho\cdot u_{,\rho}-2u/\rho^2 = \gamma^2 u_{,\tau\tau}+\varphi_{,\rho}$$

is solved using Laplace transform. In Eq. (*), ρ=radial coordinate, φ=temperature, τ=time, γ=inertia parameter depending on the velocity c of irrotational waves (all quantities dimensionless). The inverse of the solution to the transformed equation (*), which is an inhomogeneous modified Bessel equation, can be found and contains a term with the Heaviside step-function as a multiplier. If $\gamma>0$ this term (also present in the equations for stresses) reflects the presence of a shock-wave propagating with the velocity c. The remaining terms correspond to a disturbance received instantaneously throughout the medium. It appears that u depends continuously on ρ and τ, while its partial derivatives suffer discontinuities at $\tau=\gamma(\rho-1)$. Also the stresses exhibit finite jump-discontinuities on the wave front. For $\gamma>0$ a lengthy limit process carries the present solution into the corresponding quasi-static solution given earlier by the first of the authors [Nederl. Akad. Wetensch. Proc. Ser. B **60** (1957), 396–408; MR **19**, 908].

An extensive discussion based on certain numerical results is given, showing qualitative and quantitative

differences between dynamic, quasi-static and steady-state solutions.

The reviewer believes that the present paper can be counted among the classical papers on dynamic thermoelasticity. *J. Nowinski* (Madison, Wis.)

6150:

Chao, Hwei-yuan. Thermal stresses in shells of revolution of variable elastic properties. Sci. Sinica **8** (1959), 383–400.

The author considers the thermal stresses due to an axially-symmetric and steady state temperature gradient in shells of revolution in which Young's modulus, E, varies linearly with temperature. The governing differential equation is of the same form as for temperature-independent elastic properties. {Reviewer's remark: This will be so for any temperature-E law.} In searching for solutions of this equation the decrease in E is first assumed small and the resulting approximate differential equation is formally solved by an iterative method. Numerical calculations for a cylinder show that the influence of variations in E on the stresses is of the same order as the proportional variation of E. *E. H. Mansfield* (Farnborough)

STRUCTURE OF MATTER

6151:

Hauptman, H.; and Karle, J. Rational dependence and the renormalization of structure factors for phase determination. Acta Cryst. **12** (1959), 846–850.

The effect of rationally dependent atoms on phase determining formulas is described. This leads to a procedure for reinterpreting and modifying these formulas which is based upon the examination of subsets of the experimental data and a subsequent renormalization of structure factors. Although the nature of the renormalization is structure dependent, no previous structural knowledge is required for carrying out the procedure or computing phases. An example illustrating the features of rational dependence and renormalization is included.

Werner Nowacki (Bern)

FLUID MECHANICS, ACOUSTICS

See also 5813, 6136.

6152:

Ol'khovskii, I. I. A boundary value problem in generalized hydrodynamics. Soviet Physics. Dokl. **123** (3) (1958), 1164–1167 (262–265 Dokl. Akad. Nauk SSSR).

Le problème des conditions limites de l'hydrodynamique généralisée est formulé pour les problèmes unidimensionnels dans le cas d'une frontière conductrice de chaleur, en mouvement non uniforme. La fonction de distribution des vitesses étant développée en série de polynômes d'Hermite, on explicite les conditions qui font intervenir les moments jusqu'au troisième ordre compris.

J. Naze (Marseille)

6153:

Ol'khovskii, I. I. On a plane linear boundary value problem of generalized hydrodynamics (theory of the ultrasonic interferometer). Soviet Physics. Dokl. **123** (3) (1958), 1204–1207 (821–824 Dokl. Akad. Nauk SSSR).

L'auteur obtient la solution d'un problème aux valeurs limites unidimensionnel: un gaz monoatomique est situé entre deux plans parallèles infinis, dont l'un est fixe et l'autre oscillant avec une faible amplitude. Les équations hydrodynamiques utilisées sont celles qui dérivent de l'équation de Boltzmann. *J. Naze* (Marseille)

6154:

Coleman, Bernard D.; and Noll, Walter. Helical flow of general fluids. J. Appl. Phys. **30** (1959), 1508–1512.

Helical flow is obtained by superposing a Poiseuille and a Couette flow. Using the general theory of fluids proposed by W. Noll [Arch. Rational Mech. Anal. **2** (1958), 198–226; MR **21** #4596], the authors give a neat and rather thorough analysis of the steady state behavior of fluids in helical flow. Their conclusions are in accord with those of R. S. Rivlin [J. Rational Mech. Anal. **5** (1956), 179–187; MR **17**, 797], who gives a less extensive analysis of the same flows, based on a different theory.

J. L. Ericksen (Baltimore, Md.)

6155:

Nikolsky, A. A. The "second" form of motion of an ideal fluid past a solid (an investigation of discontinuous vortical flows). Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 193–196. (Russian)

6156:

Kopzon, G. I. Vibration of a thin rectangular wing of large aspect ratio in a supersonic stream. J. Appl. Math. Mech. **22** (1958), 1153–1161 (810–819 Prikl. Mat. Meh.).

The author assumes not only that the wing is thin, but also that the perturbations are small; however, it is not assumed that the motion is quasi-steady. Torsional bending deformations are considered and the normal and angular displacements are functions of time t and the spanwise coordinate z. Each deformation is expanded in a series of exponentials whose exponents are functions of z only and whose coefficients are functions of t only. A Laplace transform with respect to t is used to obtain expressions for the transforms of the coefficients. In this way stability criteria are derived. *G. N. Lance* (Dorchester)

6157:

★Richter, W. Flugmechanik. Mathematisch-Naturwissenschaftliche Bibliothek, 25. B. G. Teubner Verlagsgesellschaft, Leipzig, 1959. vi+296 pp. DM 15.60.

This is a little textbook on mechanics of flight in the tradition of the German technical colleges. It treats the familiar subjects: properties of the atmosphere; lift, drag, and moment; airplane performance, stability, and control; and closes with a descriptive chapter on drag and the boundary layer. It is not concerned with high-speed flight; thus compressibility effects are ignored.

W. R. Sears (Ithaca, N.Y.)

6158:

Racer-Ivanova, F. S. Investigation of free oscillations of a liquid of diurnal and semi-diurnal type in shallow basins. II. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 369–383. (Russian)

6159:

Dombrovsky, G. A. On the periodicity of a jet coming out of a symmetrical nozzle under rated conditions. Dokl. Akad. Nauk SSSR (N.S.) **113** (1957), 58–61. (Russian)

6160:

Voitsenia, V. S. Plane problem on oscillation of a body under two surface-separating liquids. J. Appl. Math. Mech. **22** (1958), 1121–1140 (789–803 Prikl. Mat. Meh.).

Using Kochin's method, the author considers the plane problem of wave motions induced by oscillations of a body under a surface of separation of two liquids; the upper layer has a free surface and the lower has an infinite depth. Using linearized boundary conditions and assuming the solution to be sinusoidal in time, he first obtains the explicit solution for the case of pulsating vortex and that of a source. The general problem is then reduced to consider a distribution of vortex-sources over the contour of the body. Consequently, the problem is transformed into one to solve integral equations of second kind. For small values of reduced frequency, the existence and validity of the solution as a power series in a parameter are discussed. Finally, formulas concerning the pressure and moment on the body are derived as a generalization of Kochin's and Haskind's results on one fluid.

S. S. Shu (Lafayette, Ind.)

6161:

Datta, Subhendu. A note on the motion of viscous fluid subjected to uniform or periodic body force acting for a finite time. J. Tech. Bengal Engrg. Coll. **3** (1958), 73–79.

6162:

Newman, B. G. Flow in a viscous trailing vortex. Aero. Quart. **10** (1959), 149–162.

"The equations of motion for an isolated laminar viscous vortex at moderate to large Reynolds numbers are linearized, by assuming that both the rotational velocity and the deficit of longitudinal velocity are small compared with that in the free stream. The rotational motion and the longitudinal motion may then be superimposed and solutions are readily obtained for each. If the vortex is generated by a body with profile drag it is predicted that the deficit of longitudinal velocity will be positive, which is in agreement with experimental observation. Further details of the solution and its relation to the flow in real vortices are discussed; and the theory is compared with some measurements in a turbulent vortex." (Author's summary)

G. Temple (Oxford)

6163:

Kaufmann, Walther. Die Erweiterung des Zirkulationssatzes von W. Thomson auf zähe (viskose) Flüssigkeiten. Z. Flugwiss. **7** (1959), 103–106. (English and French summaries)

6164:

Kramer, R. F.; and Lieberstein, H. M. Numerical solution of the boundary-layer equations without similarity assumptions. J. Aero/Space Sci. **26** (1959), 508–514.

The authors investigate the numerical solution of the axisymmetric compressible boundary-layer equations in the Crocco form (with x and u as independent variables). These are a parabolic system of two non-linear second-order partial differential equations, for which data can be generated on the stagnation line (at $x=0$) to serve as initial values. In the derivation of corresponding difference equations, an implicit difference scheme is used in order to avoid the stability difficulties of explicit schemes, and a special technique leads to a linear system of algebraic equations to be solved at each x-location. Numerical results for the non-dimensional shear stress and total enthalpy distributions are presented for two cases of the boundary-layer flow at high Mach numbers around a hemisphere, the first for a perfect gas, and the second with real-gas effects included. No stability analysis is given, but it is stated that the computations revealed no instability. {Two additional references may be of interest. Baxter and Flügge-Lotz [Z. Angew. Math. Phys. **9b** (1958), 81–96; MR **20** #558] use an explicit difference scheme. Rouleau and Osterle [J. Aero. Sci. **22** (1955), 249–254] apply both explicit and implicit difference schemes to incompressible boundary-layer problems with x and y as independent variables.}

D. W. Dunn (Ottawa, Ont.)

6165:

Dommett, R. L. On similar solutions of the boundary layer equations for air in dissociation equilibrium. J. Roy. Aero. Soc. **64** (1960), 36–37.

6166:

Schlichting, H. Some developments in boundary layer research in the past thirty years. J. Roy. Aero. Soc. **64** (1960), 64–80.

6167:

Finn, Robert. On steady-state solutions of the Navier-Stokes partial differential equations. Arch. Rational Mech. Anal. **3**, 381–396 (1959).

L'A. considera il problema al contorno, esterno, per il sistema di Navier Stokes stazionario, in 3 dimensioni:

$$(*)\qquad \Delta u-\nabla p = (u\cdot\nabla)u,\quad \operatorname{div} u = 0.$$

È assegnato il valore di u su certe superficie che si trovano tutte al finito; inoltre è data la condizione asintotica: $u(x)\to u_0(|x|\to\infty)$. Leray [J. Math. Pures Appl. **12** (1933), 1–82] ha dimostrato l'esistenza di una soluzione di (*) che è limite (uniforme su ogni regione limitata) di una successione $\{u_n\}$ di soluzioni di (*) che (oltre alla condizione imposta al finito) soddisfanno alla condizione $u_n(x)=u_0$ nei punti $|x|=R_n$, dove $R_n\to+\infty$. L'A. dimostra che esiste una soluzione $u(x)$ di (*) tale che $u(x)\to u_0(|x|\to\infty)$ e tale che la corrispondente funzione $p(x)$ converga a limite finito per $|x|\to\infty$.

Egli dimostra, in più, che ogni soluzione $u(x)$ di (*) definita in tutto un intorno $\mathscr{E}$ dell'infinito e tale che $\int_{\mathscr{E}}|\nabla u|^2dV<\infty$ tende a limite finito u_0 per $|x|\to\infty$. L'A.

non ottiene valutazioni asintotiche; dimostra però che, se $|u(x)-u_0|=O\ (|x|^{-\alpha})$ con $\alpha>1/2$, allora è $|u(x)-u_0|=O(|x|^{-1})$.

Le dimostrazioni si fondano principalmente su una rappresentazione integrale delle soluzioni di (*) valida nella regione illimitata $\mathscr{E}$, questa viene ottenuta utilizzando le soluzioni fondamentali introdotte da Oseen per il sistema linearizzato: $\Delta u-u_0\nabla u-\nabla p=0$, div $u=0$.

G. Prodi (Trieste)

6168:

Gershuni, G. Z.; and Zhukhovitskii, E. M. The closed convective boundary layer. Soviet Physics. Dokl. **124** (4) (1959), 102–104 (298–300 Dokl. Akad. Nauk SSSR).

In this two-dimensional steady-state problem a liquid core rotates at constant temperature ($T=0$) with constant angular velocity ω inside a circular cylinder ($r=1$) in horizontal position. The surface temperature at the lowest point is zero and varies proportionally to the sine of the arc length x. For equal thickness of temperature and momentum boundary-layers δ, profile functions of the form $u=P_1+P_2\cos 2x+\beta P_3 \sin 2x$, $T=Q_1\sin x+\alpha Q_2 \cos x$ are assumed, where P_i and Q_i are certain polynomials in y/δ. Integration of the b.l.-equations with respect to y and x yields the relations necessary for the determination of the parameters α, β, δ, and ω. This convective model is adequate for a Rayleigh number $\gg 350$.

G. Kuerti (Cleveland, Ohio)

6169:

Yih, Chia-Shun. Thermal instability of viscous fluids. Quart. Appl. Math. **17** (1959), 25–42.

The stability of a viscous, heat-conducting, fluid subjected to a downward thermal gradient is analysed for infinitesimal disturbances, when (a) the fluid lies between parallel, insulated, vertical planes, (b) the fluid is contained in a circular, vertical tube and rotates about the tube axis.

Periodicity in the vertical direction with wave number a is assumed. Both problems are governed by sixth order linear systems with two point boundary conditions. The eigen-equations are used to calculate the critical Rayleigh number R for neutral stability as a function of a. The modifying effects of rotation are also estimated.

The lowest value of the critical Rayleigh number occurs at $a=0$ for the plane boundaries case, and also for the case of axisymmetric motion in the tube with given rotation. It is likely that this is also true for the other modes of motion in the tube (these involve azimuthal variations). The most unstable modes are found to be the lowest antisymmetric ones.

Proofs are given that when there is no rotation the motion at neutral stability is time independent. Even with rotation this is true for zero wave number if the motion is axisymmetric. For the other modes of motion in a rotating tube, the motion at zero wave number is time independent if referred to a frame rotating with the tube.

A. F. Pillow (Toronto, Ont.)

6170:

DiPrima, R. C. The stability of viscous flow between rotating concentric cylinders with a pressure gradient acting round the cylinders. J. Fluid Mech. **6** (1959), 462–468.

The problem described in the title is treated within the framework of the "small gap" approximation so that the unperturbed velocity distribution consists of two parts: a linear term due to rotation and a quadratic term due to the pressure gradient. A solution is obtained by a method first used by Chandrasekhar [Mathematika **1** (1954), 5–13; MR **16**, 84] for the case in which the outer cylinder is at rest. Numerical results for the Taylor number and the wave number at the onset of instability are obtained for a range of values of the parameter $Q=3V_P/V_R$, where V_P and V_R are the average velocities of pumping and rotation respectively. Both the Taylor number and the wave number exhibit peaks in the neighborhood of $Q=-3$. [This value of Q corresponds to a situation in which, on Rayleigh's criterion, the fluid contained in the inner and outer thirds of the gap are unstable but the middle third is stable.] The results for the critical Taylor number are in remarkable agreement with the experimental results of Brewster, Grosberg, and Nissan [Proc. Roy. Soc. London. Ser. A **251** (1959), 76–91].

W. H. Reid (Providence, R.I.)

6171:

Wooding, R. A. The stability of a viscous liquid in a vertical tube containing porous material. Proc. Roy. Soc. London. Ser. A **252** (1959), 120–134.

The stability of a stratified fluid in a circular tube is studied both theoretically and experimentally. For the case in which the tube is filled with porous material, the equations of motion based on a generalization of Darcy's law are used. It is found that instability will first set in as a pattern of stationary convection which is antisymmetric about a plane through the axis of the tube and for which the wave-number in the axial direction is zero. The critical value of the density gradient is found to be $d\rho/dz=3.390\kappa\mu/gka^2$, where a is the radius of the tube, k is the permeability, and κ is the effective diffusivity. In the absence of porous material the motion is governed by the Navier-Stokes equations, and an analogous calculation for the critical density gradient gives $d\rho/dz=67.94D\mu/ga^4$, where D is the molecular diffusivity. From these two results it is then possible to obtain an expression for the ratio κ/D in terms of quantities that can be measured experimentally.

W. H. Reid (Providence, R.I.)

6172:

★Dryden, H. L. Transition from laminar to turbulent flow. Turbulent flows and heat transfer (edited by C. C. Lin), pp. 3–74. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv+549 pp. (2 plates) $15.00.

Transition is here defined as the sequence of phenomena that occur in a flow after the growth of periodic disturbances described by the linearised theory of hydrodynamic stability and that culminate in the appearance of a fully turbulent flow. In this section, the author sets out the present state of knowledge of this important but difficult problem with particular emphasis on transition phenomena in boundary layers. Although the mechanism of this transition is better understood than it was ten years ago, it is by no means clear and most of the experimental work has been directed to establishing relations between the position of the transition and the Reynolds number, the free-stream turbulence, surface roughness (distributed or local), Mach number, heat transfer, etc. A very wide selection of experimental results is presented and

correlated and appropriate reference is made to any related theoretical work. Transition in other flows (jets, mixing-layers, wakes, rotating cylinders) is described more briefly and there is a short but informative general review of the whole field.

A. A. Townsend (Cambridge, England)

6173:

★**Schubauer, G. B.; and Tchen, C. M. Turbulent flow.** Turbulent flows and heat transfer (edited by C. C. Lin), pp. 75–195. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv + 549 pp. (2 plates) $15.00.

This section is intended for readers whose primary interest is in problems arising from turbulent flow in aerodynamic contexts and, within their self-imposed restriction to mean flow properties, the authors have presented an admirable compact account of boundary layers, pipe and channel flow, and jets. Both incompressible and compressible flows are described and a very useful feature is the derivation of the momentum and energy equations for compressible flow. I believe the section itself will prove very useful to anyone approaching a problem in this field and the very full bibliography provides starting points for more specialised reading.

A. A. Townsend (Cambridge, England)

6174:

★**Lin, C. C. Statistical theories of turbulence.** Turbulent flows and heat transfer (edited by C. C. Lin), pp. 196–253. High Speed Aerodynamics and Jet Propulsion, Vol. V. Princeton University Press, Princeton, N.J., 1959. xv + 549 pp. (2 plates) $15.00.

This readable account of the theory of homogeneous turbulence is intended for the non-specialist and presents some of the essential formulation and ideas of the subject. Chapter 1, entitled "Basic concepts" includes discussion of Reynolds stresses, the forming of averages, and the probability density function of the velocity field. Chapter 2 (Mathematical formulation) commences with a development of the kinematics in real space of isotropic turbulence and a statement of the dynamical problem, followed by a rather long discussion of spectral representations. Chapter 3 (Physical aspects of the theory) contains the heart of the paper. It opens with a brief discussion of the Loitsianski integral, now of mostly historical interest, since the only interesting "result" has been found, like the big eddies, to be non-permanent. The theory of local similarity is presented carefully, and introduces a discussion of the possibility of a more extensive spectral similarity. Complete similarity is found only in the final period of decay, but Lin also examines the consequences of assuming similarity over all wave-numbers larger than the energy-containing ones, so that self-preservation is assumed for second and higher moments of the energy spectrum. An indication is given of the way that this leads to an initial period decay law. Some relevant experimental results are quoted, some of which give reasonable support for these ideas though others are, in the reviewer's opinion, indecisive. The chapter closes with a brief discussion of the use of the quasi-Gaussian relation between the fourth and second order covariances, and of the physical transfer theories, particularly that of Heisenberg. Chapter 4 (Turbulent diffusion) mentions the single-particle and two-particle analyses, together with a Lagrangian analysis by Corrsin on the development of temperature fluctuations by the turbulent convection of an initially linear temperature distribution. Chapter 5 (Other aspects) is something of an afterthought; it is little more than a list of miscellaneous problems that have been considered at one time or another.

This list shows the scope of the essay to be very broad for its 57 pages, so that the discussion is frequently rather cursory. However, there is sufficient to whet the reader's appetite and 101 references to guide him; it serves well its stated purpose as an introductory survey. The reviewer noted one unfortunate misprint on p. 227. The law of energy decay in the final period is the five-halves power law, not five-fourths as stated.

O. M. Phillips (Baltimore, Md.)

6175:

Bass, Jean. Solutions turbulentes de certaines équations aux dérivées partielles. C. R. Acad. Sci. Paris **249** (1959), 1456–1457.

The author exhibits particular solutions of M. Burger's non-linear equation in terms of functions developed from solutions of the heat equation. The auto-correlation of these functions vanishes for large values of the space or time displacement. This is interpreted as the construction of a class of "turbulent" solutions.

W. V. R. Malkus (Woods Hole, Mass.)

6176:

Popp, Simona. Corrections de compressibilité dans le problème du bilame symétrique. C. R. Acad. Sci. Paris **249** (1959), 619–621.

6177:

Storchi, Edoardo. Sulle correnti stazionarie piane dei fluidi comprimibili. Ist. Lombardo Accad. Sci. Lett. (1957/58), 349–366.

This is an extension of some previous work of Neményi and Prim [J. Math. Phys. **27** (1948), 130–135; MR **10**, 73] which shows that for steady two-dimensional flows of a perfect gas (no external forces) velocity magnitude constant along streamlines implies vorticity constant along streamlines and vice-versa, and that this implies the streamlines are concentric circles or parallel straight lines. Moreover, lines of constant velocity magnitude and constant vorticity can coincide only if they are also streamlines. The author rederives these results, in a more elegant manner, for a general compressible, non-heat conducting, inviscid fluid, showing that if any one of density, pressure, velocity magnitude, or vorticity is constant along streamlines, then the others are also constant along streamlines and the streamlines are concentric circles or parallel straight lines. Similar properties are implied by the coincidence of lines of constancy of any two and specific distinction is made between the homentropic (S = const.) case and isentropic (S = const. along streamlines) case.

P. Chiarulli (Chicago, Ill.)

6178:

Bagdoev, A. G. Investigation of the problem of penetration of pressure to the bottom of a compressible fluid. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **12** (1957), no. 3, 23–29. (Russian)

6179:

Chushkin, P. I. Subsonic flow with circulation past ellipses. Dokl. Akad. Nauk SSSR **125** (1959), 748–751 (Russian); translated as Soviet Physics. Dokl. **4**, 277–279.

6180:

Gubanov, A. I. Reflection and refraction of shock waves at the interface between two media. I. Case of normal incidence. Soviet Physics. Tech. Phys. **28** (3) (1958), 1869–1874 (2035–2040 Ž. Tehn. Fiz.).

6181:

Malyuzhinets, G. D. Sound scattering by nonuniformities in a layer of discontinuity in the sea. Soviet Physics. Acoust. **5** (5) (1959), 68–74 (70–76 Akust. Ž.).

"The scattering of sound which is propagating near a transition layer between water of different temperatures is considered when internal gravitation waves are present. Making use of the simplest assumptions concerning the static nature of the internal waves, the reverberation intensity and scattering coefficient are determined by an approximate calculation." (From the author's summary)

A. E. Heins (Ann Arbor, Mich.)

6182:

Kuan, Ting-hua. Diffraction of surface sound waves at semi-infinite impedance tubes and rods. Soviet Physics. Dokl. **124** (4) (1959), 121–124 (559–562 Dokl. Akad. Nauk SSSR).

The author discusses two problems in acoustic diffraction theory. Problem (a) deals with the solution of the partial differential equation

$$(*) \qquad \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = 0,$$

subject to the impedance boundary condition

$$\frac{\partial \Phi}{\partial r} + \sigma \Phi = 0 \quad r = a, z > 0$$

and σ a positive constant. [Conditions at infinity are not mentioned explicitly.] This problem is a standard Wiener-Hopf problem and is solved by these methods.

Problem (b) adds the boundary condition $\partial\Phi/\partial z$ assigned at the surface $z = 0$, $r \leqq a$. In this case, the problem is formulated as an infinite system of linear algebraic equations which the author states may be solved by the method of successive approximations.

A. E. Heins (Ann Arbor, Mich.)

6183:

Furduev, V. V. Interference and coherence of acoustic signals. Soviet Physics. Acoust. **5** (5) (1959), 110–115 (111–116 Akust. Ž.).

"A comparison of the properties of optical and acoustic signals with a consideration of the averaging time in connection with their reception leads to the conclusion that, unlike the optical case, the acoustic interference effect observed in speech and musical signals should vary in time, with respect to both magnitude and sign. The behavior of this variation is described by means of a running auto-correlation function with the application of an exponential weighting function. The interval of coherence of natural phonation can be determined either on the basis of integral distribution laws for instantaneous values of the running auto-correlation function, or by an investigation of the mean-square value of this function as it depends on the time of shift between signals and its repetition lag." (Author's summary)

R. A. Leibler (Princeton, N.J.)

6184:

Malkus, W. V. R. Magnetoconvection in a viscous fluid of infinite electrical conductivity. Astrophys. J. **130** (1959), 259–275.

Thermal convection is discussed in a horizontal layer with free surfaces and infinite electrical conductivity, taking into account a possible magnetic field generated by the process itself. The magnetic field is assumed to be $\mathbf{H} = c\mathbf{v}$ (satisfying $\nabla \wedge (\mathbf{v} \wedge \mathbf{H}) = 0$, where $\mathbf{v}$ is the velocity and c is a constant). The only effects of this magnetic field on the equation of motion are to give the $\mathbf{v} \cdot \nabla \mathbf{v}$ term a factor $(1 - \mu c^2/4\pi\rho)$ (where μ is the permeability and ρ the density), and to introduce a magnetic pressure. Hence one can use known solutions of the convection problem without magnetic fields, the only change being that the Prandtl number now appears divided by $(1 - \mu c^2/4\pi\rho)$. The magnitude of the magnetic field is found by varying c so as to make the mean square temperature gradient stationary, this being shown to be a necessary condition for a stable convection pattern under a variety of conditions. If the layer does not rotate, one finds that there is near equipartition of magnetic and kinetic energies, unless the Prandtl number is very large; by contrast, if the layer rotates, the ratio of magnetic to kinetic energy turns out to be infinite if the Taylor number exceeds about 4×10^4.

A. Herzenberg (Manchester)

6185:

Kogan, M. N. Magnetodynamics of plane and axisymmetric flows of a gas with infinite electrical conductivity. J. Appl. Math. Mech. **23** (1959), 92–106 (70–80 Prikl. Mat. Meh.).

The author considers flows of an ideal, perfectly conducting gas in a magnetic field which is directed parallel to the incident stream. It is shown that there are two hyperbolic flow regimes, in one of which the shock waves are inclined upstream. Simple wave solutions are presented as are solutions to problems of flows about bodies (linearized theory). Many of the results presented here have already appeared in the literature.

H. Greenspan (Cambridge, Mass.)

6186:

Cabannes, H. Dynamique des gaz ionisés: détermination des chocs stationnaires attachés à la pointe d'un dièdre. Rech. Aéro. No. 71 (1959), 3–9.

First the conservation equations for plane discontinuities (shocks) in a perfectly conducting compressible fluid are derived, following de Hoffmann and Teller [Phys. Rev. (2) **80** (1950), 692–703; MR **12**, 769] and Lüst [Z. Naturforschung **8a** (1953), 277–284; MR **15**, 271]. Three special cases are identified: (a) upstream magnetic

field normal to shock front; (b) upstream magnetic field parallel to shock front; (c) steady flow with electric field equal to zero. The last-mentioned is the case in which the velocity and magnetic vectors are parallel. This case is treated in detail, and results are presented in a series of graphs. An interesting phenomenon is the appearance, in the results, of shock angles greater than 90° for flow-deflection angles between 0° and 90°. These could be "forward-facing" shocks or shocks attached to an exterior corner (resembling the forbidden expansion shocks of conventional gasdynamics, but here being actually compression shocks). The present investigation, however, does not distinguish between these possibilities, and its author decides to omit all obtuse shock angles in plotting his final results. He expresses doubt concerning the physical reality of forward-facing shocks. It may be advisable to mention here that all shocks which violate the Second Law have been eliminated in the analysis. A general conclusion is reached that the presence of a magnetic field increases the drag (pressure rise) and decreases the temperature rise.

Finally it is argued that it is always possible to solve a steady-flow case where the electric field is different from zero by translating the coordinate system and reducing the problem to the case treated previously.

The reviewer notes that related investigations not referred to by this author include the following: H. L. Helfer, Astrophys. J. **117** (1953), 177–199 [MR **14**, 804]; J. Bazer and W. B. Ericson, ibid. **129** (1959), 758–785 [MR **21** #1806]; K. O. Friedrichs and H. Kranzer, New York Univ. Rept. **NYU-6486** (1954, reissued July, 1958).

W. R. Sears (Ithaca, N.Y.)

6187:

Kan, Uk Džon. Some cases of filtration through an earthen dam of trapezoidal cross-section on a permeable base with sloping water-support. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 3, 192–196. (Russian)

The problem mentioned in the title is studied by means of conformal-mapping theory. The ensuing integrals, which express the discharge, velocity-distribution, etc., are evaluated rigorously. Graphs or tables relating to the flow-field are not presented.

These types of problems have also been dealt with earlier by Mhitaryan [cf. Ukrain. Mat. Ž. **6** (1954), 448–456; MR **17**, 1147], to which no reference is made.

K. Bhagwandin (Oslo)

6188:

Gheorghiţă, Şt. I. Flows in harmonically nonhomogeneous porous media. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) **2** (**50**) (1958), 19–26.

The author presents a formal mathematical solution to the problem of the Darcy-type flow in harmonically non-homogeneous porous media. In the general case, the filtration-coefficient too satisfies a Helmholtz-type equation. Boundary-conditions are stated and the arbitrary constants which appear in the ensuing expansions are also evaluated explicitly.

Numerical results are not presented. It seems, however, that these problems could be studied more effectively by means of Mathieu-Hill functions, etc.

K. Bhagwandin (Oslo)

OPTICS, ELECTROMAGNETIC THEORY, CIRCUITS

See also 5798, 5802.

6189:

Maroni, Pascal. Phénomènes de décharge dans les plasmas lorentziens: étude de la distribution électronique en présence d'un champ magnétique. C. R. Acad. Sci. Paris **249** (1959), 881–883.

6190:

Maroni, Pascal. Phénomènes de décharge dans les plasmas lorentziens. Étude de la distribution électronique dans le champ magnétique. C. R. Acad. Sci. Paris **249** (1959), 914–916.

6191:

Hettner, G.; und Wagner, H. Fourier-Analyse des elektrischen Mikrofeldes in einem Plasma. I. Ann. Physik (7) 4 (1959), 89–95.

Quadratische und ähnliche Mittelwerte der in einem verdünnten Plasma auftretenden elektrischen Feldintensitäten wurden schon öfters berechnet. Ziel der vorliegenden Untersuchung ist dagegen, den zeitlichen Verlauf von solchen Feldstärkekomponenten statistisch zu bestimmen. Zu diesem Zwecke wird die Fourier-Transformierte der Feldintensität $\mathfrak{E}_x(t)$

$$(1)\qquad F_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathfrak{E}_x(t)e^{i\omega t}dt$$

berechnet. Das Plasma wird so stark verdünnt angenommen, dass man die Coulombschen Wechselwirkungen der Teilchen vernachlässigen kann, und ausserdem wird nur das Feld der (sich schnell bewegenden) Elektronen (dagegen nicht das der Ionen) berücksichtigt. Durch Einführung von Polarkoordinaten für den Ort und die Geschwindigkeit eines herausgegriffenen Elektrons zerfällt (1)—wenn man die Feldstärke mit Hilfe dieser Koordinaten ausdrückt—in zwei Integrale, welche durch die modifizierten Besselschen Funktionen dritter Art K_0 und K_1 ausgedrückt werden können. Weiter werden dann über alle N Elektronen summiert und das erhaltene F_ω in einen reellen und einen imaginären Teil zerlegt. Um die weiteren Rechnungen durchführen zu können, wird erstens angenommen, dass alle Elektronengeschwindigkeiten einfach gleich der wahrscheinlichsten gaskinetischen Geschwindigkeit $\sqrt{(2kT_e/m)}$ (T_e=Elektronentemperatur) sind und zweitens, dass das Feld eines Elektrons infolge der Abschirmung auf den sogenannten Debye-Radius $D=\sqrt{(kT_e/4\pi nq^2)}$ (wo n die Elektronendichte und q die Ladung bedeuten) beschränkt ist.

Endlich folgt unter Benützung eines Satzes von A. Markoff für die Wahrscheinlichkeit, dass $|F_\omega|$ zwischen $|F_\omega^0|$ und $|F_\omega^0|+d|F_\omega^0|$ liegt

$$(2)\qquad W(\omega, |F_\omega^0|)d|F_\omega^0| = \frac{|F_\omega^0|}{2\Phi(\omega)}\exp\left(-\frac{|F_\omega^0|^2}{4\Phi(\omega)}\right)d|F_\omega^0|,$$

also eine Gauss-Verteilung, wie das ja auch zu erwarten war. ($\Phi(\omega)$ wird mit Hilfe von Bessel-Funktionen dritter Art ausgedrückt.) Ausserdem wird auch noch der wahrscheinlichste Wert von $|F_\omega^0|$ berechnet und die erhaltenen

Resultate werden auch graphisch dargestellt. Die Berücksichtigung der Coulombschen Wechselwirkungen wird den Gegenstand einer weiteren Veröffentlichung bilden.

T. Neugebauer (Budapest)

6192:

Neufeld, Jacob. Occurrence of Vavilov-Čerenkov radiation in a high-temperature plasma. Phys. Rev. (2) **116** (1959), 1–3.

"Akhiezer and Sitenko investigated the energy loss of a charged particle moving through a plasma at a velocity considerably lower than the mean thermal velocity of the electrons in the plasma and determined the component of the stopping power due to the transverse electric field produced by the plasma and acting upon the particle. The presence of such a component may in some instances be associated with the occurrence of the Vavilov-Čerenkov radiation. It is shown, however, that in this particular case the field surrounding the particle decreases very rapidly with the distance from the particle and no radiation takes place." *Author's summary*

6193:

Mower, Lyman. Conductivity of a warm plasma. Phys. Rev. (2) **116** (1959), 16–18.

"A theory for obtaining the conductivity of a uniform plasma as a function of frequency and temperature is presented and compared with a number of recent treatments." *Author's summary*

6194:

Bellman, Richard; and Kalaba, Robert. Functional equations, wave propagation and invariant imbedding. J. Math. Mech. **8** (1959), 683–704.

The subject of this paper was discussed before by the same authors [Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 317–319; see MR **20** #5030]. In the present paper, however, the necessary analysis has been worked out in much more detail. The general idea, the "localization principle", concerns the possibility of deriving the change of a function (representing a physical property) inside an infinitesimal space interval as if the medium were homogeneous beyond it. This general principle is illustrated for a one-dimensional wave propagation according to the equation $d^2u/dx^2+k^2(x)u=0$. A wave incoming from $x=-\infty$ is split in each infinitesimal layer of thickness dx into a reflected wave (internal reflection) moving back towards $x=-\infty$, and a transmitted wave continuing in the original direction. The amplitudes of these waves are independent of the properties of the medium outside the interval dx. This splitting into two waves is repeated ad infinitum for all waves produced after any number of internal reflections. The addition of all contributions resulting after $2N$ reflections leads to a wave $U_{2N}(x)$ travelling towards $x=+\infty$; that of the other contributions connected with $2N+1$ reflections involves a wave $u_{2N+1}(x)$ travelling towards $x=-\infty$. The complete wave function can be represented by $\sum_0^\infty u_{2N}(x)+\sum_0^\infty u_{2N+1}(x)$, as shown by the present reviewer [*The theory of electromagnetic waves, a Symposium*, pp. 169–179, Interscience Publishers, New York, 1951]. The rigorous validity of this expansion is shown here under a sufficient condition, which is very general.

The wave propagation under consideration is connected with a reflection coefficient $c_1(z)$ defined as follows. Let us replace the spaces $x<z$ and $x>b$ by homogeneous media characterized by the wave numbers $k(z)$ and k_2, respectively. The coefficient $c_1(z)$ then represents the ratio at $x=z$ of the total reflected wave $\sum_0^\infty u_{2N+1}(x)$ (continuing in $x<z$ towards $x=-\infty$), and of the original wave $u_0(x)$ (moving in $x<z$ towards $x=+\infty$) producing the former. The classical method leads to a Riccati equation for this coefficient. According to the "localization principle" the same equation is arrived at by considering exclusively reflections at the boundaries of a single stratum of thickness dx, while taking into account the only three contributions that are of the first order with respect to an infinitesimally decreasing thickness.

Similar considerations are applied to a non-uniform transmission line terminated by a load. The impedance $Z(x)$ of the line, as observed at a distance x from the load (while looking towards the latter), shows properties analogous to those of the former reflection coefficient $c_1(z)$. Both quantities are independent of the medium beyond a special boundary; both satisfy a Riccati equation. Finally, the general concept of waves transmitted through and reflected internally by an inhomogeneous medium (extending beyond a special boundary) has also been worked out for a single linear differential equation of any order N, and for a set of linear differential equations of the second order.

{A serious printing error occurs on page 692; c_2 in equation (2) should read c_1.} *H. Bremmer* (Eindhoven)

6195:

Ávila, Geraldo Severo de Souza. Simultaneous propagation of waves of more than one type. Notas Mat. No. 15 (1959), vii+28 pp.

6196:

Ginzburg, V. L.; and Eidman, V. Ia. Čerenkov radiation from dipoles. Soviet Physics. JETP **35**(8) (1959), 1055–1058 (1508–1512 Ž. Eksper. Teoret. Fiz.).

The authors derive the radiation losses experienced by electric and magnetic dipoles moving in a continuous medium. They find that the result depends on whether one considers the magnetic dipoles as a pair of magnetic monopoles, or as a circulating electric current. They also discuss the radiation losses if the moments move in a channel or a slit. *N. L. Balazs* (Princeton, N.J.)

6197:

Maslov, V. P. An asymptotic expression for the eigenfunctions of the equation $\Delta u+k^2u=0$ with boundary conditions on equidistant curves and the propagation of electromagnetic waves in a waveguide. Soviet Physics. Dokl. **123** (3) (1958), 1132–1135 (631–633 Dokl. Akad. Nauk SSSR).

The title of this paper describes its contents. The author considers a two-dimensional region, the reduced scalar wave equation $\Delta\psi_k+k^2\psi_k=0$, and the boundary condition that the solution ψ_k vanishes on the equidistant curves which represent the boundary. His result is that $\psi_k(r,s)-\sin(\pi nr/a)z(s)\ \{(1-K(s)r)(1-K(s)a)\}^{-1/2}=0(a\gamma/n)$. Here r and s are coordinates specially chosen to represent any point in the guide, a is the width of the guide, $K(s)$

represents the curvature of a wall of the guide along which s is arc length, and $\gamma=\{k^2-(mn/a)^2\}^{1/2}$. For fixed k the result is asymptotic in n; for fixed n it is asymptotic for large γ or large k. When the equidistant curves are straight lines the results are interpreted geometrically in terms of rays. *M. Kline* (New York, N.Y.)

6198:

Kuznecov, A. V. Non-repeating contact schemes and non-repeating superpositions of functions of algebra of logic. Trudy Mat. Inst. Steklov. **51** (1958), 186–225. (Russian)

The author investigates contact schemes, applying the usual ideas of graphs and algebra of logic. In the graphs considered, more than one edge may exist between some of the vertices. A connected graph whose k vertices, called poles, are significant is called a (k-poles) network. A subgraph S of a network N which is a network itself is called a subnetwork of N if its poles are exactly the poles of N contained in S and the end-vertices of edges of $N \dot{-} S$. The most important case, that of 2-poles-networks, is considered. Such a network is reducible if it contains a non-trivial subnetwork (with two poles); otherwise it is irreducible. A network N is called a scheme if a variable of algebra of logic (a. of l.) corresponds to each edge of N. Thus a function of a. of l. called conductibility may be defined for every scheme S: this function is equal to 1 (or 0), according to whether the poles of S both belong (or not) to the same component of the subgraph of S, which consists of all vertices of S and of those edges of S with the value 1. A scheme is called non-repeating if different variables correspond to different edges. A function $\Phi(x_1, \cdots, x_n)$ of a. of l. is reducible if it can be obtained as a non-repeating superposition, i.e., if it can be written in the form $\Phi=\Psi(x_{i_1}, \cdots, x_{i_s}, \Theta(x_{j_1}, \cdots, x_{j_r}))$, where $x_{i_1}, \cdots, x_{i_s}, x_{j_1}, \cdots, x_{j_r}$ is some permutation of the variables $x_1, \cdots, x_n$. It is proved that, in the case of non-repeating schemes, reducible functions correspond to reducible schemes and conversely. Further, superpositions of functions of a. of l. are studied. These functions are divided into four classes: parallel, serial (mod 2)-functions and others, which are shown to be mutually disjoint. The decomposition of a function into irreducible functions by means of non-repeating superpositions is proved to be unique. The author shows that a. of l. is not powerful enough, even if extended by a finite number of conjunctions, to be used for investigating schemes which are neither parallel, nor serial ones. Thus, no "universal bridge scheme" exists. The author suggests using the algebra of relations [Birkhoff, *Lattice theory*, Amer. Math. Soc., New York, 1948; MR **10**, 673] for similar investigations of contact schemes. In the supplement, all irreducible networks with less than 10 edges are listed. *I. Friš* (Prague)

6199:

Trahtenbrot, B. A. The theory of non-repeating contact schemes. Trudy Mat. Inst. Steklov. **51** (1958), 226–269. (Russian)

The article deals with the same problems as the preceding one by Kuznecov. Two schemes are called equivalent if the same function corresponds to both of them, and isomorphic if they are isomorphic as graphs so that the edges equally denoted correspond to each other and the poles correspond to poles. Obviously isomorphic schemes are equivalent. For irreducible networks the converse holds, i.e., equivalence implies isomorphism. Further, reducible networks are discussed. The notion of a canonical decomposition is introduced, which is proved to be unique. Functions as well as schemes may be divided into three classes: parallel, serial and others. It is proved that for non-repeating schemes serial functions are realized by serial schemes and similarly for the two other classes. A scheme obtained from a scheme Ψ by replacing an edge of Ψ by a scheme Φ, so that the poles of Φ and the new ones of Ψ are identified, is called a superposition of Φ into Ψ. That can be done (for a given edge) in two different ways. The operation changing the scheme obtained by one such way into the other one is called a rotation. Since every reducible scheme can be considered as obtained by superposition, this operation may be applied to every reducible scheme. The theorem that each two equivalent schemes can be made isomorphic by a finite number of rotations is proved. Finally, the synthesis of non-repeating schemes is discussed. A method to construct a non-repeating scheme to a given function is shown. This method is successful whenever such a scheme exists. *I. Friš* (Prague)

6200:

Saltzer, Charles. Algebraic topological methods for contact network analysis and synthesis. Quart. Appl. Math. **17** (1959), 173–183.

The author defines a state function with respect to a given pair of nodes in a network as the formal product of the coboundaries of all the zero cochains (coefficient modulo two) for which one node has coefficient one and the other zero. This is applied to the treatment of the analysis and synthesis problems for contact networks and is also related to the Boolean admittance function for the pair of nodes. *J. B. Giever* (University Park, N.M.)

6201:

★Aseltine, John A. Transform method in linear system analysis. McGraw-Hill Electrical and Electronic Engineering Series. McGraw-Hill Book Co., Inc., New York-Toronto-London, 1958. xvi+300 pp. $8.50.

The book gives a general treatment of linear transform methods in the analysis of physical systems. (I) More than half of the exposition (118 pages) is devoted to Laplace transform methods (elementary and general properties of the transform; inversion methods by partial fraction expansions and calculus of residues and by complex inversion formula). Applications include lumped electrical and mechanical systems, feedback analysis, and continuous system problems of the theory of mechanical vibrations, heat conduction, and transmission lines. Different approaches to the problem of analysis, initial conditions and switching problems, and the concept of system function are discussed. (II) Subsequently properties of Fourier series and Fourier transforms are discussed. Random processes are introduced and described by probability functions, preparatory to a treatment of systems with random inputs by Fourier methods (55 pages). (III) Ladder networks and more general systems represented by difference equations are discussed. The solution of difference equations is obtained by means of

the Z-transform (13 pages). (IV) A short summary of general integral transforms, with emphasis on the Mellin transform concludes the exposition (7 pages). An appendix contains elements of complex variable theory and a list of Laplace transforms.

The exposition emphasizes physical and engineering aspects and tries to convey the physical significance of the many steps involved in the mathematical formalism. {To this reviewer such a tentative attempt to restore the role of intuition in mathematics appears particularly attractive because for engineers and physicists intuition might be a safer guide in the solution of complex problems than the methods in which intuition and common sense are replaced by convergence criteria.} *B. Gross* (Rio de Janeiro)

6202:

Bottani, Ercole; e Sartori, Rinaldo. Tensioni e correnti in una rete di resistori contenente un solo generatore. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 498–505.

A proof is presented of the intuitive fact that, in a resistive network containing one single generator, the voltage across the generator is larger than any other branch voltage. This is done with the aid of the theorem stating that in a closed, not necessarily linear, network with no energy storage the sum of the powers absorbed or generated in the branches is zero. It is shown that the potential of a node of only dissipative branches is intermediate between the potentials of the neighboring nodes. From this fact the desired proof is deduced. Further it follows that no branch current exceeds that of the generator branch. *H. A. Haus* (Vienna)

6203:

Hahn, Stefan. The instantaneous complex frequency concept and its application to the analysis of the building up of oscillations in oscillators. Proc. Vibration Problems **1959**, no. 1, 29–47. (Polish and Russian summaries)

If $\psi(t)$ is a complex time signal of relatively slowly varying frequency and amplitude, the logarithmic derivative $p(t)=\dot\psi/\psi$ represents an instantaneous complex frequency. Its usefulness in the treatment of classical self-excitation problems is illustrated by application to the simplest van der Pol-equation. The paper is expository and covers mainly old ground.

H. G. Baerwald (Albuquerque, N.M.)

6204:

★Tuttle, David F., Jr. Network synthesis. Vol. 1. John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London; 1958. xv+1175 pp. $23.50.

This weighty book is but the initial volume in a two-volume treatment of the synthesis of passive electrical networks. That so much can be written on just one phase of electric circuit theory is an indication of how many new developments have taken place in this area during the past thirty years.

The present volume is concerned in its entirety with the synthesis of two-terminal RLC network functions and related topics. The preliminary material dealing with the basic concepts of network theory is presented in an informal fashion, with no attempt at completeness and precision. This is a little disconcerting to a reader expecting a uniformly high level of exposition in a work of this length.

Few contributions have influenced the development of network synthesis as much as Brune's classical work [J. Math. Phys. **10** (1931), 191–236] on positive real functions, which play a key role in the theory of passive two-terminal RLC networks. The author devotes more than one hundred pages to the discussion of positive reality and the energy relations from which this notion is derived. Here one finds a variety of tests for determining whether a given rational function is positive real, with each test illustrated by a carefully worked out example. The determination of the immittance function from the knowledge of its real part or imaginary part or magnitude or phase is discussed at great length in Chapter 8. Yet the author fails to note the interesting observation made by Guillemin [Proc. Nat. Electronics Confer. **9** (1954), 513–532; MR **17**, 1030] concerning the possibility of associating delta-function type real parts with reactance functions, which throws much light on the relationship between the real and imaginary parts of an immittance function.

The core of the book (Chapters 9 and 10) constitutes a very thorough and illuminating exposition of the classical methods of Brune, Bott-Duffin, and Darlington for the synthesis of positive-real immittance functions. The more recent method of Miyata [J. Inst. Elec. Engrs. Japan **35** (1952), 211–218] and its extension by Kuh [Trans. I.R.E. **CT-2** (1955), 302–308; MR **19**, 94] are treated less completely, while Pantell's technique [Proc. I.R.E. **42** (1954), 861] is discussed but the clarifying observation due to Storer [ibid., 1451] is not mentioned except in the bibliography.

The chapters on approximation and potential analogy (Chapters 13 and 14) contain a great deal of material which is not available in other texts. The treatment of approximation, though, is along conventional lines, with the emphasis placed on least squares, Taylor, Padé and Chebyshev approximations. The bulk of new material appears in the discussion of potential analogies and their applications, reflecting the substantial contributions made to this subject by the author and his students.

The influence of the Guillemin school is in evidence throughout the text, both in regard to the choice of topics and the way in which they are treated. The author's informal style makes for easy reading, but in some instances it also makes it difficult to extract from the text a precise statement of a theorem or a definition of a new term. Despite its minor shortcomings, this book is likely to remain for a long time the standard reference on the subject of two-terminal networks.

L. A. Zadeh (Berkeley, Calif.)

6205:

Borskiĭ, V. On properties of impulsive responses of varying-parameter systems. Avtomat. i Telemeh. **20** (1959), 848–855. (Russian. English summary)

This paper is devoted to theorems characterizing the impulse response $W(t,\tau)$ of a physical system governed by an equation

$$(*)\qquad \sum_{i=0}^{n} a_i(t)u^{(i)}(t) = \sum_{j=0}^{m} b_j(t)v^{(j)}(t) \quad (m < n),$$

where $a_i(t)$, and $b_j(t)$ are sufficiently regular functions, and $v(t)$ and $u(t)$ are, respectively, the input and output of the system. The function W has the property that, for arbitrary u and v satisfying (*), $u(t)=\int_{-\infty}^{t} W(t, \tau)v(\tau)d\tau$. Uniqueness theorems characterizing the possible equations (*) with given $W(t, \tau)$ are also obtained. The results supplement and correct previous work in this direction by Batkov [Avtomat. i Telemeh. **19** (1958), 49–54; MR **19**, 1149].
E. Reich (Minneapolis, Minn.)

CLASSICAL THERMODYNAMICS, HEAT TRANSFER

See also 6169.

6206:

Pustovoit, S. P. Transient thermal convection in a spherical cavity. J. Appl. Math. Mech. **22** (1958), 800–806 (568–572 Prikl. Mat. Meh.).

Le problème considéré est le suivant : à l'instant $t=0$ un liquide au repos, à température constante T_0, remplit une sphère dont la paroi est maintenue à la température constante $T_1 < T_0$.

L'auteur étudie le régime transitoire en convection naturelle qui prend naissance à $t=0$. Le nombre de Grashof G étant supposé petit, la solution est développée suivant une série de ce petit paramètre. L'approximation d'ordre zéro (transfert par conduction sans convection) étant connue, l'approximation d'ordre un est calculée sous forme d'un développement en série de fonctions propres de $\Delta\phi(r)+\lambda\phi(r)=0$, $\phi(1)=0$. Les lignes de courant résultant de cette première approximation, ainsi que les isothermes, ont été déterminées dans un exemple numérique.
R. Gerber (Grenoble)

6207:

Schellenberger, Günter. Lösung der eindimensionalen Wärmeleitungsgleichung mit willkürlicher Wärmequellenverteilung. Acta Hydrophys. **5** (1959), 201–212.

Die Lösung wird mit Hilfe der Laplace-Transformation in bekannter Weise formal in der Form mehrfacher Integrale angeschrieben. Irgend ein Fortschritt ist damit wohl kaum erzielt.
H. Parkus (Vienna)

6208:

Sevruk, I. G. Laminar convection over a linear heat source. J. Appl. Math. Mech. **22** (1958), 807–812 (573–576 Prikl. Mat. Meh.).

Le problème étudié est celui de la détermination du régime de convection naturelle créé dans un fluide en milieu infini, par un fil chauffant horizontal. Le problème est traité avec les approximations de la couche limite laminaire. La détermination des lignes de courant et des isothermes est ainsi réduite à la résolution d'une équation différentielle non linéaire, pour laquelle la solution est donnée sous forme d'une série dont les premiers coefficients sont explicités. Comme l'auteur le fait remarquer, l'hypothèse que le fil soit infiniment mince, et le fait que le mouvement devienne turbulent à une certaine distance du fil, limitent l'application des résultats.
R. Gerber (Grenoble)

6209:

Dennis, S. C. R.; Mercer, A. McD.; and Poots, G. Forced heat convection in laminar flow through rectangular ducts. Quart. Appl. Math. **17** (1959), 285–297.

Le problème considéré est celui de la détermination du nombre de Nusselt local pour l'écoulement d'un fluide visqueux incompressible, en régime laminaire établi, dans un canal cylindrique, lorsque le fluide, qui entre à température constante, échange de la chaleur le long du canal, avec un milieu ambiant à température constante.

La méthode de résolution du système des équations aux dérivées partielles, posées par ce problème, est analysée dans le cas d'une section de canal quelconque. La méthode utilise les développements en séries de fonctions propres.

Les formules résolutives sont explicitées lorsque le canal est de section rectangulaire. Dans le cas particulier où la paroi est à température constante, les valeurs numériques des trois premières valeurs propres et fonctions propres sont données, pour différents rapports des deux dimensions de la section.

Dans le cas où il y a une condition d'échange à la paroi ($\partial\theta/\partial n+N\theta=0$ en variables réduites) les premiers termes du développement de la température moyenne, et du Nusselt, ont été calculés seulement pour un canal carré. Ils sont donnés pour $N=2$, 10, et 20.

Les résultats sont comparés avec ceux de S. H. Clark and W. M. Kays [Trans. A.S.M.E. **75** (1953), 859–866].
R. Gerber (Grenoble)

6210:

Nakabe, Kazunobu. On the general solution for the two-dimensional problems of steady heat conduction in the domain with an arbitrary form. J. Osaka Inst. Sci. Tech. Part I **8** (1958), no. 2, 83–91.

A Fourier series is used as the solution of two dimensional Laplace equation. Let the boundary conditions be written as $G_i(r, \theta)=G_i\{f_i(\theta), \theta\}=g_i(\theta)$, $(i=1, 2)$; the Fourier coefficients are then readily obtained by direct substitutions. It is also suggested that, when the inner boundary is a circle, an approximation may be obtained by taking finite terms of the series to satisfy conditions at finite points of the boundary. Numerical examples of regular polygons with circular cut-out and zero temperature at the boundaries are given by means of approximation.
L. N. Tao (Chicago, Ill.)

6211:

Shchelkin, K. I. Two cases of unstable combustion. Soviet Physics. JETP **36** (**9**) (1959), 416–420 (600–606 Ž. Eksper. Teoret. Fiz.).

Es werden folgende Formen einer instabilen Verbrennung betrachtet: (1) Die Instabilität der Flammenfront in einer ebenen Detonationswelle; (2) die Instabilität der Zündzone als Quelle hochfrequenter Schwingungen in Raketenbrennkammern. Für beide Phänomene werden die Stabilitätskriterien angegeben. Für das zweite Problem wird auch noch der maximale Druck abgeschätzt, der bei den Schwingungsvorgängen auftreten kann.
L. Speidel (Mülheim)

QUANTUM MECHANICS

See also 6120.

6212:

Winogradzki, Judith. Sur le principe variationnel des champs spinoriels. C. R. Acad. Sci. Paris **249** (1959), 911–913.

6213:

★Glauber, R. J. High-energy collision theory. Lectures in theoretical physics, Vol. I. Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1958 (edited by W. E. Brittin and L. G. Dunham), pp. 315–414. Interscience Publishers, New York-London, 1959. vii+414 pp. $6.00.

A systematic development of high-energy scattering theory, and its applications. The optical model and diffraction processes are discussed, along with such specialized applications as the "eclipse" effect in deuterium. The lectures bring together a great deal of work that had not before been published in one place.

H. W. Lewis (Madison, Wis.)

6214:

Engelmann, F.; und Fick, E. Die Zeit in der Quantenmechanik. Nuovo Cimento (10) **12** (1959), supplemento, 63–72.

The authors point out that in quantum mechanics "time" is used in two different ways. On the one hand it denotes the topological sequence of events and hence is a c-number in the theory. On the other hand, time also figures in the description of the dynamical behavior of the system, and as such is a physical observable which then corresponds to a Hermitian operator. This operator is defined and its properties discussed. With the aid of the time operator one can easily prove the uncertainty relation between energy and time.

M. J. Moravcsik (Livermore, Calif.)

6215:

Fubini, S.; and Walecka, D. Dispersion analysis of possible parity nonconservation in low-energy pion-nucleon scattering. Phys. Rev. (2) **116** (1959), 194–202.

"An attempt is made to analyze the possible nonconservation in low-energy pion-nucleon scattering. The use of relativistic dispersion relations enables us to relate such effects to the possible parity nonconservation in strange-particle production from pion-nucleon collisions. We show that large violations of parity conservation in strange-particle production are indeed compatible with small effects in low-energy pion-nucleon scattering. It is suggested that our result might be useful in order to understand the very good evidence for parity conservation in nuclear physics."

Authors' summary

6216:

Maksimov, L. A. On the scattering matrix in an indefinite metric. Soviet Physics. JETP **36 (9)** (1959), 324–329 (465–473 Ž. Eksper. Teoret. Fiz.).

Developing an idea of Heisenberg, Bogolyubov, Medvedev and Polivanov [Nauč. Dokl. Vysš. Školy. Ser. Fiz.-Mat. no. 1 (1958)] showed that a theory with an indefinite metric can give physically reasonable results if one adds to the initial states of the physical system a definite amplitude of the non-physical states and imposes the standing wave condition upon the amplitude of the non-physical states. This method, however, dispenses with microscopic causality. The present work gives an alternative method which does not have this disadvantage. The actual S-matrix for the physical states is defined by $S'=U^{-1}SU$, where S is the conventional scattering matrix and U is a pseudo-unitary matrix which produces a transformation of S to a form in which the matrix elements between physical and non-physical states are zero.

The general condition is found which permits the existence of such a transformation U, and it is shown how one then can determine U. The method is illustrated on the example of the Lee model and the scalar photon model. In conclusion it is shown that in order to obtain physically reasonable results, it is necessary to give up either microscopic causality or the exactly unitary character of the S' matrix. However, "in the large" S' is always both causal and unitary.

P. Roman (Manchester)

6217:

Budini, P.; and Furlan, G. Electron-positron elastic scattering from extended nuclei. Nuovo Cimento (10) **13** (1959), 790–801. (Italian summary)

6218:

Kazes, E. Generalization of the Levinson-Jauch theorem to an arbitrary number of channels. Nuovo Cimento (10) **13** (1959), 983–987. (Italian summary)

"For a system having a finite number of channels the bound states of the free and complete Hamiltonian are shown to be simply related to the determinant and trace of the S-matrix."

Author's summary

6219:

Hanke, L. Delbrückstreuung am homogenen Magnetfeld. Acta Phys. Austriaca **12** (1958/59), 472–474.

With the help of a Green's function for electrons (and positrons) in a homogeneous magnetic field constructed previously [L. Hanke and P. Urban, same Acta **12** (1958/59), 304–314; MR **20** #7518], the authors show that in the lowest approximation the Delbrück scattering of a photon will vanish.

P. G. Bergmann (New York, N.Y.)

6220:

★Соколов, А. А. Введение в квантовую электродинамику. [**Sokolov, A. A. Introduction to quantum electrodynamics.**] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 534 pp. 18.05 rubles.

The present book is essentially an improved and considerably enlarged version of the author's previous work, the first part of *Kvantovaya teoriya polya* [Sokolov and Ivanenko, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; MR **14**, 1044], which was later translated into German and edited by Akademie Verlag, Berlin, in 1957, under the title *Quantenelektrodynamik* [MR **19**, 922]. Its aim is to give a comprehensive account of the quantum theory of interactions between photons and

electrons, although related questions, such as electromagnetic interactions of the muons, and the two-component theory of the neutrino, are also discussed.

The author employs a unified mathematical treatment, based on the Heisenberg picture, and uses what he calls a direct method of calculations. This consists essentially in applying operational calculus techniques, Green's functions and Fourier methods. The application of the now so familiar Dyson-Feynman techniques is avoided. Renormalization theory is not treated, although various vacuum processes are discussed at some length. In these cases simple regularization methods are used.

In many processes and phenomena which are treated, a historical approach is stressed: the process is first described in the framework of classical and non-relativistic theory, then according to elementary quantum mechanics, and finally, from the point of view of relativistic quantum electrodynamics.

The first chapter gives a detailed account of the general quantum theory of free fields, including methods of solving the field equations, and goes into considerable detail in connection with Dirac's equation. The second chapter discusses the interaction of electrons with the quantized electromagnetic field, first in general terms, then applying these methods at length to concrete processes. The following chapter deals with the theory of the positron; while the last one deals essentially with vacuum-fluctuation and vacuum-polarization phenomena of the electron-positron field, although a considerable amount of more classical material also finds its place in this chapter.

The presentation is, as a whole, clear and delightful, even though sometimes rather characteristic of the personal attitude of the author. The printing and the appearance of the work are fine, and the price, as is the usual case with Russian books, is very moderate.

P. Roman (Manchester)

6221:

Shirokov, Yu. M. Relativistic corrections to phenomenological Hamiltonians. Soviet Physics. JETP **36** (9) (1959), 330–332 (474–477 Ž. Eksper. Teoret. Fiz.).

It is shown that starting from the general postulate of relativistic invariance and using group theoretical considerations and the properties of the S-matrix, any given phenomenological non-relativistic Hamiltonian describing the interaction between particles of arbitrary mass and spin can be corrected relativistically in a consistent fashion, at least up to terms of order $(v/c)^2$.

P. Roman (Manchester)

6222:

Ida, Masakuni. On the relation between the phase shift and the number of bound states. Progr. Theoret. Phys. **21** (1959), 625–639.

The author discusses the Low equation for a particular model investigated earlier by F. J. Dyson [Phys. Rev. (2) **106** (1957), 157–159]. A knowledge of the position and properties of the bound states in the model allows a unique solution of the Low equation to be selected. This is done with the aid of a certain relation between the scattering phase shift and the number of bound states. This relation is a generalization of earlier results by Levinson [Danske Vid. Selsk. Mat.-Fys. Medd. **25** (1949), no. 9; MR **10**, 710], Jauch [Helv. Phys. Acta **30** (1957), 143–156; MR **19**, 498], Haag [Nuovo Cimento (10) **5** (1957), 203–215] and others.

G. Källén (Lund)

6223:

Jauch, J. M.; and Zinnes, I. I. The asymptotic condition for simple scattering systems. Nuovo Cimento (10) **11** (1959), 553–567. (Italian summary)

This paper is concerned with the characterization of those interactions in quantum field theory which permit the definition of an S matrix. This problem is equivalent to obtaining conditions for the existence of the limit of $V_t{}^*U_t$ as t approaches plus or minus infinity, where $U_t = \exp(-iH_0t)$, $V_t = \exp(-iHt)$, H_0 is the free particle energy and H is the total energy of the system. Necessary and sufficient conditions for this are obtained in two forms. These are applied to a central potential giving a sufficient condition which is weaker than that previously given by J. M. Cook [J. Math. and Phys. **36** (1957), 82–87; MR **19**, 1011].

A. J. Coleman (Kingston, Ont.)

6224:

Liotta, R. S. Further developments of the covariant functional formalism for interacting fields. III. Nuovo Cimento (10) **13** (1959), 921–935. (Italian summary)

The author develops a formalism of quantum field theory by replacing the usual Hamiltonian by an "invariant Hamiltonian", which is defined as the trace of the energy-momentum tensor. In this way he attempts to remove the surface-dependent terms from the interaction energy density in the interaction representation for certain interacting fields, but his treatment becomes even more complicated than the usual one.

S. N. Gupta (Detroit, Mich.)

6225:

Longoni, A. M. Teoria non locale dell'effetto Compton. Nuovo Cimento (10) **13** (1959), 802–808. (English summary)

"Ci si propone di applicare al caso dell'effetto Compton la teoria non locale proposta da G. Wataghin e di studiare il comportamento asintotico della sezione d'urto differenziale secondo tale teoria."

Riassunto dell'autore

6226:

Low, F. E. Boson-fermion scattering in the Heisenberg representation. Phys. Rev. (2) **97** (1955), 1392–1398.

"It is shown that the S-matrix for boson-fermion scattering can be simply expressed in the Heisenberg representation. By performing a time integration one obtains the S-matrix in the Schrödinger representation, which has the same form as the conventional perturbation theory sum over s states. Suitably limiting the nature of the intermediate states entering into this sum leads to integral equations for certain matrix elements which are equal to the S-matrix elements on the energy shell. These equations appear in a completely renormalized form. For example, in the fixed source limit, the four pion-nucleon scattering states satisfy the same equation (with different numerical coefficients). The equations are nonlinear, but involve only the scattering phase shifts. The equivalent equation for photopion production is linear, and in the

fixed source limit can be written down from a knowledge of the experimental scattering phase shifts. The zero-pion-mass theorems of Gell-Mann and Goldberger (concerning the isotopic spin independence of the zero-energy S-wave scattering) and of Kroll and Ruderman [same Rev. **93** (1954), 233–238] follow simply from the formalism." (Author's summary)

The essential results of this paper have since been obtained by more perspicuous methods, e.g., by G. C. Wick [Rev. Mod. Phys. **27** (1955), 339–362; MR **17**, 334]. As a method of calculation, they have played a decisive role in obtaining the consequences of the fixed source meson theories [G. F. Chew and F. E. Low, Phys. Rev. (2) **101** (1956), 1570–1579; MR **21** #1858]. The fundamental significance of the integral equations of this method was furthermore clarified by a proof [Castillejo, Dalitz, and Dyson, ibid. **101** (1956), 453–458] that they do not have unique solutions. They thus characterize a class of theories rather than a unique dynamical theory.

A. Klein (Philadelphia, Pa.)

6227:

Novozhilov, Iu. V. On the scattering of "dressed" particles in quantum field theory. Soviet Physics. JETP **35** (8) (1959), 515–520 (742–749 Ž. Eksper. Teoret. Fiz.).

This paper develops a scattering formalism for quantum field theory in which the asymptotic description of the particles takes fully into account their self-interactions. The asymptotic states of many particles are represented as suitably symmetrized products of exact single particle state-vectors which are one-particle eigenvectors of the full Hamiltonian. The formalism, which in one direction develops the work of Low [paper reviewed above] and in another that of H. Ekstein [Nuovo Cimento (10) **4** (1956), 1017–1058; MR **18**, 626], works completely in terms of matrix elements of operators between such asymptotic states. The construction of both scattering amplitudes and potentials, when the latter concept is applicable, is considered. A similar method has been developed by R. E. Cutkowsky [Phys. Rev. (2) **112** (1958), 1027–1038; MR **20** #5665]. None of this work takes into account vacuum polarization phenomena. In this sense it is less satisfactory than existing Lorentz covariant techniques, and, moreover, solves no problems that these are unable to handle.

A. Klein (Philadelphia, Pa.)

6228:

★Wataghin, G. Multiple production of mesons and a nonlocal theory of fields. Lectures in theoretical physics, Vol. I. Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1958 (edited by W. E. Brittin and L. G. Dunham), pp. 204–237. Interscience Publishers, New York-London, 1959. vii+414 pp. $6.00.

A discussion of multiple meson production. After a clear and detailed description of the present state of the phenomenology of the high-energy cosmic ray jets, some of the theoretical approaches are described. The statistical model of Fermi is described; then the author's own nonlocal theory, which is illustrated by applying a relativistically covariant cut-off in the Lee model. This leads to a choice of cut-off function which is then applied to a specific theory of multiple meson production.

H. W. Lewis (Madison, Wis.)

6229:

Fronsdal, C. A generally relativistic field theory. Nuovo Cimento (10) **13** (1959), 988–1006. (Italian summary)

This paper introduces the frankly speculative idea that physics should deal with four-dimensional manifolds that are locally imbeddable in a six-dimensional flat space. The author points out that many physically interesting solutions of Einstein's field equations are in fact so imbeddable; he further justifies his proposal by the argument that manifolds imbeddable in a S_6 represent a lesser generalization of special relativity than those imbeddable in a S_{10} (the most general four-dimensional V_4). {Reviewer's comment: Such an argument of "simplicity" of a physical theory depends, of course, on one's viewpoint. If structure of the invariance group is adopted as the criterion of "simplicity", then Fronsdal's field theory represents a complication of general relativity.} The author then proceeds to develop possible action principles that would characterize physical theories inscribed on these imbedded V_4 space-time manifolds. He considers combinations of the metric with (a priori unspecified) other fields. The Lagrangian, which from the point of view of the S_6 must be an antisymmetric tensor density of rank 2, is subjected to the following restrictions: (1) The non-metric variables enter with no higher than first derivatives; (2) the Lagrangian is independent of the imbedded V_4; (3) the coefficient in the bilinear term in a Schwinger-type action principle is to possess certain symmetry properties.

P. G. Bergmann (New York, N.Y.)

6230:

Misra, S. P. Action principle and Lagrangian with higher order derivatives. Indian J. Phys. **33** (1959), 461–468.

"We have obtained here certain commutation relations of field quantities (field operators and their derivatives) for space-like separation of the points when the Lagrangian density contains derivatives of field operators of order higher than first by using Schwinger's operator principle of Stationary Action. The commutators thus obtained are quite complicated, and the consistency of this procedure can only be discussed in individual cases."

Author's summary

6231:

Gold, Louis. Wave mechanics for hydrogen atom in cylindrical coordinates: non-separable eigensolutions. J. Franklin Inst. **268** (1959), 118–121.

One way for a mathematician to irritate at least some of his colleagues is first to state the solution of a problem and then to show that it satisfies all conditions. It sets one puzzling how the author came to that specific answer.

In the present paper the author starts with the wave equation for the H-atom and introduces dimensionless cylindrical coordinates. The resulting equation in two variables is non-separable. The author then states: "Careful study reveals that solutions of the form $\rho^n \exp[\beta(\rho^2+n^2)^{1/2}]$ are appropriate $\cdots$" and the reviewer finds himself in the mental state described above. What kind of reasoning led the author to infer this type of solution for this particular non-separable equation? The answer to this question might provide indications for the solution of other non-separable equations.

As for the present equation, it possesses solutions of the more general form $\rho^n \exp[\beta(\rho^2+u^2)^{1/2}] \sum a_i(u/\rho)^i$, but the reviewer derived this from the well-known solutions in spherical coordinates. The references could be supplemented with J. Brillouin, C. R. Acad. Sci. Paris **229** (1949), 513–514 [MR **11**, 281], in which a non-separable equation for Sommerfeld's half-plane diffraction problem is solved.
D. J. Hofsommer (Amsterdam)

6232:

Shaw, Gordon L. The imaginary part of the optical model potential for neutron interactions with nuclei. Ann. Physics 8 (1959), 509–550.

Many of the earlier calculations of the imaginary part of the optical model potential from the nucleon-nucleon interaction have given contradictory results in poor agreement with experiment. This paper makes a thorough examination of the problem on the basis of perturbation theory. Calculations are made both with smooth two-particle potentials, such as the Yukawa and the exponential potentials, and also with the G matrix calculated from a realistic potential by Brueckner and his collaborators. Both the real and imaginary parts of the optical model potential are treated as functions only of the local nucleon density. The results show that the imaginary part increases as the density decreases from its value in the center of the nucleus, and reaches a maximum of twice this value when the density is about 20 per cent of the maximum density. Absorption of neutrons should therefore occur mainly in the surface of the nucleus, and there is experimental evidence to support this theory. Some correction terms are discussed, and evidence is produced to show that they are quite important, possibly doubling the imaginary part of the potential, but no satisfactory numerical calculation is described. The compound nucleus is discussed, and it is suggested that narrower resonances due to two-particle states might occur as well as the broad one-particle resonances.
D. J. Thouless (Birmingham)

6233:

Layzer, David. On a screening theory of atomic spectra. Ann. Physics 8 (1959), 271–296.

A theory is presented for the description of atomic properties which is essentially a perturbation scheme in which the zero-order eigenfunctions are strictly hydrogenic. Thus, all eigenfunctions corresponding to selected principal quantum numbers are taken into account in first order, rather than those corresponding to selected principal and azimuthal quantum numbers. The theory provides a natural explanation of Moseley's law and the screening doublet law. Suitably developed it should be successful for the prediction of atomic properties which are insensitive to the detailed nature of the wave functions, but it is open to question whether it would be successful in predicting, say, the photoionization cross section of potassium, which is anomalously small.
A. Dalgarno (Belfast)

6234:

Verlet, Loup. Contribution à l'étude du modèle optique du noyau. Ann. Physique 4 (1959), 643–687.

Après avoir montré que l'on pouvait reproduire les sections efficaces de diffusion élastique et d'absorption pour les neutrons par un potentiel complexe remplaçant l'interaction véritable du nucléon au sein de la matière nucléaire, l'auteur essaye de prévoir les caractéristiques de ce potentiel (potentiel optique) à partir des interactions fondamentales nucléon-nucléon. Le calcul du potentiel optique se simplifie beaucoup si l'on décrit le noyau par un gaz de Fermi infini et si l'on suppose que, dans les états intermédiaires, les particules subissent un potentiel moyen constant (indépendant de leur énergie). Les erreurs introduits par ces deux approximations se compensent sans doute.

Pour que le développement en approximation de Born converge, l'auteur est amené à utiliser un potentiel nucléon-nucléon qui a une forme radiale régulière. Ceci ne permet pas, alors de reproduire avec exactitude les sections efficaces aux hautes énergies puisque cette hypothèse exclue ainsi les potentiels à cœur dur.

En utilisant un potentiel gaussien central à forme radiale unique, on rend assez bien compte du potentiel réel en fonction de l'énergie on reproduit très bien la variation du potentiel imaginaire avec l'énergie, et on obtient $-13{,}5$ MeV pour l'énergie de volume au lieu des -16 MeV donnés par l'expérience. Finalement, l'auteur étudie la convergence de la série de perturbation et montre que le principe de Pauli est essentiel pour assurer cette convergence qui paraît bonne.

La réussite de cette tentative ne doit pas, comme le fait remarquer l'auteur, dissimuler le fait que le potentiel optique calculé est obtenu à partir d'interactions fondamentales pas très réalistes.
P. Chevallier (Strasbourg)

6235:

Takagi, Shuji; Watari, Wataro; and Yasuno, Masaru. Note on the spin-orbit coupling and tensor forces. Progr. Theoret. Phys. **22** (1959), 549–565.

"The spin-orbit coupling in heavy nuclei is investigated on the basis of strong tensor interaction between nucleons, characteristic of the pion theory of nuclear force. It is shown that the process which includes the exchange of the incident nucleon with those of the target nucleus deformed by itself is important in the low energy region, and that this process provides the spin-orbit coupling with a correct sign and reasonable magnitude."
Authors' summary

6236:

★Peierls, R. E. Selected topics in nuclear theory. Lectures in theoretical physics, Vol. I. Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1958 (edited by W. E. Brittin and L. G. Dunham), pp. 238–314. Interscience Publishers, New York-London, 1959. vii+414 pp. $6.00.

A very clear survey of some topics of considerable current interest in nuclear theory. Among the topics treated are the many-body problem and its application to the derivation of the shell model, the optical model of nuclear reactions, and the nuclear collective degrees of freedom. The treatment is lucid and authoritative throughout.
H. W. Lewis (Madison, Wis.)

6237:

Jancovici, B. Spin-orbit coupling and tensor forces. Progr. Theoret. Phys. **22** (1959), 585–594.

"The possible explanation of the spin-orbit coupling, in

heavy nuclei, by second-order effects of the tensor forces is investigated. The various second order terms are described by four graphs, in the case of one particle outside closed shells. The main contribution comes from an exclusion effect: two particles of the closed shells cannot, by mutual excitation, jump into the orbit which is already occupied by the outside particle. This effect could account, at least partially, for the observed spin-orbit splitting. The case of a hole is also investigated, and shows the same kind of agreement. A comparison with other works is discussed." *Author's summary*

6238:

★Stech, Berthold. Strange particles and their interactions. Lectures in theoretical physics, Vol. I. Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1958 (edited by W. E. Brittin and L. G. Dunham), pp. 82–119. Interscience Publishers, New York-London, 1959. vii+414 pp. $6.00.

A systematic exposition of the main facts (both experimental and theoretical) about the strange particles. The treatment is simple and clear, although intentionally far from complete. It can serve as a useful primer for this field.

H. W. Lewis (Madison, Wis.)

6239:

★Rohrlich, F. Pair production and bremsstrahlung in the field of an atom. Lectures in theoretical physics, Vol. I. Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1958 (edited by W. E. Brittin and L. G. Dunham), pp. 1–29. Interscience Publishers, New York-London, 1959. vii+414 pp. $6.00.

A clear systematic treatment of the pair-production and bremsstrahlung phenomena in the field of an atom. The theory is described and all the theoretical corrections are reviewed critically. Finally, some comparison with experiment is made, but only after the reader is briefed on how seriously to take the theory.

H. W. Lewis (Madison, Wis.)

6240:

Nakamura, Hiroshi. Global symmetry and a connection in iso-space. Progr. Theoret. Phys. **21** (1959), 827–836.

This paper considers an extended isospin group in which the law of transformation for isospinors and isovectors is assumed to be

$$(1)\qquad \psi'(x+dx)=(1+g\partial_\mu\boldsymbol{\phi}\cdot\boldsymbol{\tau}\gamma_5dx^\mu)\psi(x),$$

$$(2)\qquad \boldsymbol{\phi}'(x+dx)=\boldsymbol{\phi}(x).$$

Here ψ and $\boldsymbol{\phi}$ are isospinors and isovectors to be associated with the baryon and pi meson fields, $\boldsymbol{\tau}$ and γ_5 are the conventional isospin and Dirac pseudoscalar matrices, g is a constant of dimensions length, and $\partial_\mu=\partial/\partial x^\mu$. Requiring the Lagrangian to be invariant under this group leads to the pseudovector form of meson-baryon coupling

$$ig\partial_\mu\boldsymbol{\phi}\cdot\bar{\psi}\gamma_\mu\gamma_5\boldsymbol{\tau}\psi.$$

This result is interesting since previous attempts to derive the form of meson-nucleon coupling by requiring invariance under a group depending upon a function of space-time always led to a coupling of baryons to vector rather than to pseudoscalar mesons. The author recognizes that the law of transformation (2) is not what would be obtained from (1) by relating the transformation law for isovectors to that for isospinors, but feels that these two need not be related in the conventional way. This novel point of view is, however, not gone into.

S. Bludman (Berkeley, Calif.)

6241:

★Good, R. H., Jr. Theory of particles with zero rest-mass. Lectures in theoretical physics, Vol. I. Lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1958 (edited by W. E. Brittin and L. G. Dunham), pp. 30–81. Interscience Publishers, New York-London, 1959. vii+414 pp. $6.00.

A special treatment is given of particles with zero rest-mass, and arbitrary spin greater than zero. The treatment is illustrated on the spinor (neutrino) field, and the vector (electromagnetic) field. The quantization is then discussed, and the connection between spin and statistics is derived. *H. W. Lewis* (Madison, Wis.)

6242:

Buccafurri, A.; and Fano, G. Formulae for Feynman graphs of arbitrary topology. Nuovo Cimento (10) **13** (1959), 628–636. (Italian summary)

"A compact formula is given, which collects together for any perturbative order, large numbers of topologically equivalent graphs. Problems of enumeration and classification are notably simplified by its use." (Authors' summary) *S. Deser* (Waltham, Mass.)

6243:

Butkov, E. Spin-orbit potential in pseudoscalar theory. Nuovo Cimento (10) **13** (1959), 809–817. (Italian summary)

6244:

Dürr, H.-P.; Heisenberg, W.; Mitter, H.; Schlieder, S.; und Yamazaki, K. Zur Theorie der Elementarteilchen. Z. Naturf. **14a** (1959), 441–485.

About two years ago rumours were circulating among physicists to the effect that a new theory of elementary particles had been invented which was able to explain all the difficult problems in this field. Statements were even quoted saying that the construction of big accelerators could be abandoned as one should now be able to handle everything theoretically in a satisfactory way. Some of the excitement even penetrated to the daily press. A preprint containing a sketch of some of these ideas was also circulated on a limited scale but soon withdrawn by one of the authors involved.

Since then, the theory has been discussed and rather severely criticised at various conferences but no comprehensive account of the material has been available so far. The paper to be reviewed here contains the background and actual facts of all this excitement. The new theory is

built on the differential equation

$$\gamma_\nu \frac{\partial}{\partial x_\nu} \psi \pm C^2 \gamma_\mu \gamma_5 \psi (\bar{\psi} \gamma_\mu \gamma_5 \psi) = 0.$$

The first part of the paper contains a discussion of the group-theoretical properties of this equation. First of all, the theory is invariant under Lorentz transformations but also under the so-called Pauli-Pursey transformation $\psi \to a\psi + b\gamma_5 C^{-1}\bar{\psi}^T$ [W. Pauli, Nuovo Cimento (10) **6** (1957), 204–205; MR **19**, 612; D. L. Pursey, ibid. **6** (1957), 266–277; MR **19**, 1129]. It has been remarked by F. Gürsey [ibid. **7** (1958), 411–415] that this group is isomorphic to the rotation group in a three-dimensional space. Therefore, the invariance of the fundamental equation above under this transformation can be used to construct an isotopic spin space in the theory. The equation is also invariant under the transformation $\psi \to \exp[i\alpha\gamma_5]\psi$. This is used to construct another quantum number which the authors suggest should be identified with the baryonic number. Unfortunately, these invariance properties of the theory are not enough to explain all the conservation laws one has in nature as there is no place for the leptonic quantum number. It is then remarked that the above equation also has another "invariance property", viz., that it is unchanged by a scale transformation, where all lengths in the theory are multiplied by a certain factor and all masses by the inverse of the same number. It is suggested, that this property can be exploited to construct a new quantum number in the theory. Unfortunately the reviewer is unable to give a satisfactory explanation of exactly how this is done.

Another large part of the paper is concerned with detailed numerical computations of the mass eigenvalues of the various states in the model. Basically, the authors use the Tamm-Dancoff-method for this purpose. The numerical values of the masses obtained are not too different from the masses of physical particles, but the reliability of the computation is very hard to judge.

The interaction between various particles in the model is also discussed with the aid of the same mathematical tools. The big problem here is to explain why all of these interactions are not invariant under rotations in isotopic spin space in spite of the fact that fundamental equation of motion in the model has this invariance property. The authors point out that there exists an analogous situation in elementary relativistic quantum mechanics. The Dirac equation for the hydrogen atom is invariant under three-dimensional rotations but there exists no solution to this equation which is exactly invariant under such rotations. The two S-states making up the ground state of the hydrogen atom are only rotational-invariant in a non-relativistic approximation but transform into each other in the exact theory. It is suggested that the model of elementary particles discussed by the authors has a similar structure.

The paper ends with a discussion of some mathematical questions of principle. Some space is devoted to a discussion of the Lee model and the equations of that model are rewritten in an integro-differential form. The purpose of this rewriting is to get an equation where everything is finite and mathematically well defined. In particular, the authors try to avoid the handling of the product of two field operators at the same time. A product of this kind is replaced by an integral over a small time interval. Finally, arguments are given to explain why the authors believe that non-linear theories have weaker singularities on the light cone than linear theories. *G. Källén* (Lund)

6245:

Ishida, Shin. Remarks on the final state interaction in Fermi's theory of multiple particle production. Progr. Theoret. Phys. **22** (1959), 207–212.

"A method of modifying Fermi's statistical formula of multiple particle production is presented. It consists of expressing the 'matrix element' of Fermi's theory as an integration of plane wave in the effective interaction volume, and then replacing this free state wave function by that of interacting particles. Its application to the π-N system and deuteron production is given. Belenky's formalism on the same problem is critically discussed."

Author's summary

6246:

★Marshak, Robert E. Meson physics. Dover Publications, Inc., New York, 1959. viii + 378 pp. $1.95.

This book is a reprint of the original 1952 McGraw-Hill edition. Some of the topics discussed are: production of pions in photo-nucleon and nucleon-nucleon collisions, capture and absorption of negative pions, general properties of pions and muons, and pion-nuclear interactions. The theoretical analyses are mostly perturbation field theoretical calculations and phenomenological ones. Much of this is stated rather than derived, giving the book a descriptive character. The ravages of time have, of course, taken their toll and hence the advances of the past eight years have made much of both the theoretical and experimental discussions out of date and incomplete. However, the book still contains numerous items of interest to a student in this field.

R. Arnowitt (Syracuse, N.Y.)

6247:

Rojansky, V. *Q*-number modification of the Lorentz rotations. Phys. Rev. (2) **114** (1959), 634–635.

The author proposes an eight-component Dirac-equation, obtained by replacing the usual mass term $mc^2\beta$ by $mc^2(a\beta + b_1\pi_3\beta + b_2\pi_3)$, $\pi_1\ \pi_2\ \pi_3$ being a set of anticommuting square roots of unity, commuting with the Dirac matrices α and σ. The equation so obtained is irreducible in general; the rest energies are $\pm(a \pm (b_1^2 + b_2^2)^{1/2})mc^2$, and Lorentz invariance requires that the transformation matrix $S = \exp(\frac{1}{2}\theta\alpha_x)$ be replaced by $\exp(\frac{1}{2}\theta\pi_3\alpha_x)$, etc. The author speculates that such an equation may be used to represent mass doublets (like electron-μ-meson), and in general Fermions whose magnetic g-factor differ from 2.

F. Villars (Cambridge, Mass.)

6248:

Fulton, Thomas; and Schwed, Philip. Elastic nucleon-deuteron scattering in a soluble model as a test of the impulse approximation. Phys. Rev. (2) **115** (1959), 973–979.

One of the main tools for studying scattering or production reactions from a particle which is part of a nucleus is the impulse approximation. The validity and accuracy of this approximation has always been an important target of investigations. The present paper treats a somewhat idealized situation in which the impulse approximation and the Born approximation can be compared directly. The model is that of neutron-deuteron scattering in which the two-body interaction is completely described by the effective range theory. It is shown that, (a) the impulse

approximation and the Born approximation differ considerably; (b) the pick-up term is not treated adequately by either approximation; (c) the off-energy-shell effects are significant. *M. J. Moravcsik* (Livermore, Calif.)

6249:

Jepsen, Donald W.; and Hirschfelder, Joseph O. Set of co-ordinate systems which diagonalize the kinetic energy of relative motion. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 249–256.

A general procedure is described, which is shown to yield coordinates for an n-particle problem in terms of which the kinetic energy is a sum of squares. The coordinates are components of vectors joining the centers of mass of certain subsets of the system of particles.

W. Kaplan (Ann Arbor, Mich.)

6250:

Sawada, K.; Brueckner, K. A.; Fukuda, N.; and Brout, R. Correlation energy of an electron gas at high density. Plasma oscillations. Phys. Rev. (2) **108** (1957), 507–514.

This work is closely related to two previous publications on the subject [M. Gell-Mann and K. Brueckner, same Rev. **106** (1957), 364–368; MR **19**, 98; and K. Sawada, ibid., 372–383; MR **19**, 98] and is concerned with the contribution of the zero point energy of plasma oscillations to the correlation energy in a high density electron gas. This contribution is included in the result of the first publication, but not separated from the contributions of the pair states (scattering states). It is shown that the plasma modes are stable below the continuum of particle excitations, and their contribution to the correlation energy is evaluated. Finally it is shown that the selective summation of the perturbation series by Gell-Mann and Brueckner, and the use of the "truncated" Hamiltonian commutators by Sawada corresponds to the random phase approximation of Bohm and Pines [D. Bohm and D. Pines, ibid. **92** (1953), 609–625]. *F. Villars* (Cambridge, Mass.)

6251:

Brout, R. Correlation energy of a high-density gas. Plasma coordinates. Phys. Rev. (2) **108** (1957), 515–517.

This is a sequel to #6250 above and is, in the main, an investigation of the completeness of the eigenstates of the Sawada Hamiltonian for the dense electron gas [K. Sawada, op. cit., #6250]. It is shown that the inclusion of plasma modes (plasma oscillation) is necessary to guarantee the completeness of the set. Explicit expressions for both scattering and plasma modes of excitation are given.

F. Villars (Cambridge, Mass.)

6252:

Amado, R. D.; and Brueckner, K. A. Moment of inertia of interacting many-body fermion systems. Phys. Rev. (2) **115** (1959), 778–784.

This paper deals with the effect of interparticle forces on the moment of inertia $\mathscr{I}$ of a rotating system of Fermions in its ground state. Rotation is expressed by imposing periodic boundary conditions in a rotating coordinate system. It is first shown that $\mathscr{I}$ has the rigid body value in the absence of interparticle forces; due to the identity of particles this result may indeed be interpreted as due to rigid rotation. The first order effect of interaction is then shown to be zero; this is due to an exact cancellation of two effects of the interaction: modification of the effective mass of the particles and change of the value of the matrix elements appearing in the formula for $\mathscr{I}$.

This result has been generalized in a recent paper by Rockmore [Phys. Rev. (2) **116** (1959), 469–474].

F. Villars (Cambridge, Mass.)

RELATIVITY.

See also 6221.

6253:

★Ueno, Yoshio. Axiomatic method and theory of relativity equivalent observers and special principle of relativity. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 322–332. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi + 488 pp. $12.00.

In the first part of the paper, the concept of equivalent observers is investigated by the axiomatic method. This part is a short review of the contents of an earlier paper by Y. Ueno and H. Takeno [Progr. Theoret. Phys. **8** (1952), 291–301; MR **14**, 506]. The second part is devoted to the axiomatization of Einstein's special principle of relativity. The notions of equivalent observers, space-frame, time-frame, and uniform motion are used as the fundamental concepts. The author's aim was not so much to achieve logical exactness as to investigate all possible ways of expressing the contents of the special principle of relativity. *S. Bażański* (Warsaw)

6254:

Kooy, J. M. J. On relativistic rocket mechanics. Astronaut. Acta **4** (1958), 31–58.

6255:

Nijboer, B. R. A.; and Groenewold, H. J. The so-called clock-paradox in relativity theory. Nederl. Tijdschr. Natuurk. **25** (1959), 160–180. (Dutch)

This is an exceedingly clear review of the clock paradox and the debates which grew up around it. The authors discuss in detail how the effect arises, using space-time diagrams both from the point of view of the clock which is at rest in an inertial frame, and from the point of view of the clock which suffers accelerations. The experimental verification of the effect is also discussed, and an exhaustive set of references is given.

N. L. Balazs (Princeton, N.J.)

6256:

Schmutzer, Ernst. Beitrag zur Geometrisierung der klassischen Feldphysik. Wiss. Z. Friedrich-Schiller-Univ. Jena **8** (1958/59), 15–30.

The present, somewhat extensive article represents an amplification and extension of an earlier paper by the author [Z. Physik **149** (1957), 329–339; MR **19**, 927], in

which the fundamentally new idea consists of a generalization of the geometry of 5-dimensional Riemannian space according to which the line-element $d\vartheta = e_\nu dx^\nu$ need no longer represent an exact differential. The application of this theory to cosmology was studied by the author in a subsequent paper [Ann. Physik (7) **1** (1958), 136–144; MR **19**, 1239]. The present article begins with a detailed exposition of the underlying geometrical concepts, and it is emphasized that the lack of holonomicity results from the anholonomicity of the structure of the local coordinate frames (bases), while only holonomic coordinate transformations are permitted. According to the author this geometry is suitable for the geometrical unification of the gravitational field, the electromagnetic field and a new field which appears in the course of his developments, the electromagnetic field resulting from the lack of exactness of $d\vartheta$. The connection of this theory with the projective theory of relativity is established by means of projective specialization. A number of new points of view result from the physical interpretation of the theory: these are largely concerned with the field equations, the equations of motion, and the possibility of including meson fields. Interesting comparisons are made with the work of other authors, in particular as regards 4- and 5-dimensional mass densities [A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme*, Masson, Paris, 1955; MR **17**, 199] and the inclusion of material fields [G. Ludwig, *Fortschritte der projektiven Relativitätstheorie*, Vieweg, Braunschweig, 1951; MR **14**, 213]. Unfortunately it is not possible to give an adequate description of the author's great variety of results within the limits of a review.
H. Rund (Durban)

6257:

Husain, Saiyid Izhar. Sur les discontinuités des tenseurs de courbure en théorie unitaire d'Einstein. C. R. Acad. Sci. Paris **246** (1958), 3020–3022.

En utilisant la méthode introduite par Lichnerowicz pour l'étude de la radiation gravitationnelle l'A. établit les équations vérifiées par le tenseur discontinuité du tenseur de courbure en théorie unitaire asymétrique. Ce sont des relations du type (pour le "système fort"):

$$l_\nu H^{\alpha}_{\beta,\lambda\mu} + l_\mu H^{\alpha}_{\beta,\nu\lambda} + l_\lambda H^{\alpha}_{\beta,\mu\nu} = 0$$

$$l_\alpha H^{\alpha}_{\beta,\lambda\mu} = 0$$

$$l^\beta H^{\alpha}_{\beta,\lambda\mu} = 0$$

(l_α normale à l'hypersurface de discontinuités) qu'il étudie, de façon générale, pour un tenseur $H_{\alpha\beta,\lambda\mu}$ possédant les symétries du tenseur de courbure.
Y. Fourès-Bruhat (Reims)

6258:

Husain, Saiyid Izhar. Sur les discontinuités des tenseurs de courbure en théorie unitaire d'Einstein (système faible des équations du champ). C. R. Acad. Sci. Paris **248** (1959), 194–196.

Cette note est la suite d'une étude de l'auteur sur les discontinuités des tenseurs de courbure et de Ricci dans le cas du système fort [vide sup]. Les discontinuités du tenseur de Ricci s'expriment sous la même forme dans les deux cas, à l'aide de deux vecteurs orthogonaux l_α, n_β, mais les identités satisfaites par le tenseur de courbure comportent un second membre fonction de n_α, l_β, qui n'existe pas dans le cas du système fort où seulement l_α intervient dans ces identités.
J. Renaudie (Rennes)

6259:

Lenoir, Marcel. Identités de conservation en théorie du champ unifié. C. R. Acad. Sci. Paris **249** (1959), 44–46.

In a previous paper [same C. R. **248** (1959), 1944–1946; MR **21** #2508a] the author proposed a generalization of the unified field theory of Einstein and Schrödinger. He now derives the Bianchi identities for this generalization. He applies these to the equations of Bonnor's unified field theory [Proc. Roy. Soc. London. Ser. A **226** (1954), 366–377; MR **16**, 755] and questions the compatibility of these equations.
W. B. Bonnor (London)

6260:

Toupin, R. A. World invariant kinematics. Arch. Rational Mech. Anal. **1** (1958), 181–211.

The author constructs a formalism for the treatment of different kinematics on a parallel basis, making easier a comparison of classical and relativistic kinematical theories. Kinematical quantities are expressed in terms of space-time (world) coordinates, demanding the expressions to be invariant under coordinate transformations of the Euclidean, Galilean and Lorentzian group, respectively. Each of these groups determines a corresponding 4-dimensional geometric space with its kinematics as a theory of invariants of a motion in the corresponding space.

To construct the formalism, use is made of the Klein's principle [F. Klein: *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen, 1872] assuming that Euclidean, Galilean and Lorentzian groups of transformations are subgroups of the group G_A of unrestricted analytic transformations on all four general coordinates of events. The subgroups are referred to the preferable coordinates in the corresponding spaces, in terms of which fields of geometric objects have canonical forms, invariant with respect to the G_A. All three kinematical systems are formulated as theories of joint invariants (under G_A) of a motion and of a set of corresponding fields.

As an illustration of the world invariant formalism introduced, some fundamental invariants of motion (velocity, measure of deformation) are defined in Euclidean, Galilean and Lorentzian kinematics of a continuous medium. The author emphasizes the fact that in the case of Lorentzian kinematics it is impossible to separate rigid motions from the general owing to the open question of a proper definition of rigid motions in the theory of relativity.

The question of convected coordinates in Euclidean and Galilean kinematics is considered separately, as well as the invariance problem in the 3-dimensional space of material points under a group G_M of allowable material coordinate transformations. Invariance of the constitutive equations is discussed as an example.

At the end of the paper, the kinematical quantities are interpreted as two-point fields [T. Doyle and J. L. Ericksen, *Nonlinear elasticity*, Academic Press, New York, 1956; MR **17**, 915; R. A. Toupin, J. Rational Mech. Anal. **5** (1956), 849–915; MR **18**, 349], invariant under two groups

of transformations: the first is a 4-dimensional group G_A of transformations of events and the second is a 3-dimensional group G_M of transformations of material coordinates.
T. P. Anđelić (Belgrade)

6261:
Mohorovičić, Stjepan. Über die Möglichkeit auch anderer spezieller Relativitätstheorien. Methodos **10** (1958), 267–286.

The author's new "special theories of relativity" appear to be of the following kind. For two systems S, S' moving relative to each other in the direction of the x and x' axes with constant velocity v, the proportionality of the intervals: $x'^2+y'^2+z'^2-c^2t'^2=\lambda(x^2+y^2+z^2-c^2t^2)$ is assumed, the transformations being such that x', y', z' are linear functions of t and x, y, z respectively, while t' is linear in x, y, z, t. The coefficients of these transformations do not appear to be related to v. In particular, writing $x'=\alpha x-\beta t$, the author expresses all other coefficients as functions of α and β, which he insists are arbitrary {Reviewer's note: $\lambda=(\alpha^2-\beta^2/c^2)$.} His new theories are obtained by assigning certain values to α and β, and, on the strength of certain philosophical arguments, the criterion of agreement with experimental observations is largely ignored, although some physical consequences are briefly discussed.
H. Rund (Durban)

6262:
Brulin, O.; and Hjalmars, S. The gravitational zero mass limit of spin-2 particles. Ark. Fys. **16** (1959), 19–32.

This work generalizes an earlier paper [Ark. Fys. **14** (1958), 49–60; MR **20** #3014] by the same authors on the obtaining of a relativistic spin-2 equation in a Dirac equation formalism. In the present paper a rather general mass term is considered which in the mass zero limit gives the linearized Einstein equations for the gravitational field. The Dirac-type wave function $\psi^{\alpha\beta\gamma}$ can be made proportional to the Freud superpotential.
S. Bludman (Berkeley, Calif.)

6263:
Bopp, Fritz. Bemerkungen zur Konforminvarianz der Elektrodynamik und der Grundgleichungen der Dynamik. Ann. Physik (7) 4 (1959), 96–102.

This paper gives an especially perspicacious proof of the conformal invariance of Maxwell's equations, and shows that conformal invariance of the equations of motion would require that the mass transform as a reciprocal length. [For the Dirac equation this latter point was made by J. A. Schouten and J. Haantjes in Akad. Wetensch. Amsterdam Proc. **39** (1936), 27]. {The conformal transformations have for so long attracted physicists because they constitute a simple group containing the inhomogeneous Lorentz transformations, because they are admitted by the equations of pure electrodynamics, and because they raise such tantalizing questions about mass or length. The reviewer is struck by the fact that all authors discuss only the invariance group of the differential equations without studying the formulation of boundary and initial value problems and questions of causality. The answers to these questions are important in any physical application and would illuminate the physical implications of the nonlinear and singular nature of the conformal transformations.}
S. Bludman (Berkeley, Calif.)

ASTRONOMY

See also 6081.

6264:
Lindblad, Bertil. On the dynamics of the central layer in the galaxy. Stockholms Obs. Ann. **20** (1958), no. 6, 18 pp.

As in recent earlier papers [same Ann. **18** (1955), no. 6; **19** (1956), no. 7; **19** (1957), no. 9; **20** (1958), no. 4; MR **17**, 419; **19**, 227], the author considers dispersion orbits, that is, orbits in the galactic plane along which the matter of a stellar association will be spread out in the course of the time. In a certain coordinate system with the rotational speed ω', the dispersion orbit coincides with the motion of a particle. As a new kind of orbit special orbits are introduced as an approximation to the dispersion orbits. The rotational speed ω' has, namely, been modified to ω_s', so that closed orbits are obtained.

Resonance phenomena, which occur between the frequency of the disturbing force and the radial frequency, are examined in section 2 of the paper. After that two special cases are investigated; namely, in section 3, sectorial harmonic waves in a spheroidal central layer, adhering closely to the angular motion of the apsidal lines of special orbits, and, in section 4, a rotating elongated ring of matter for small masses coinciding with a dispersion orbit or a special orbit. Elliptical rings rotating with the speed of the circular motion can give rise to bars in the spirals. In this respect the theory differs only formally from that set forth by Lindblad and Langebartel [ibid. **17** (1953), no. 6; MR **15**, 356].
E. Lyttkens (Uppsala)

6265:
★Grémillard, Jean. Recherches sur les conditions d'existence de solutions périodiques de la troisième sorte du problème des trois corps. Thèses, Université Paris. Gauthier-Villars, Paris, 1959. iii+112 pp.

This paper contains a thorough discussion of the complicated question of periodic solutions of the third kind for the three-body problem. The method follows closely the ideas of Poincaré and Zeipel; however, the discussion is not restricted to small inclinations. If p/q (p, q relatively prime integers) is the ratio of the periods of the osculating periods. The treatment is very different for even and odd values of $q-p$, at least for larger values of the inclination. Previous announcements appeared in C. R. Acad. Sci. Paris **244** (1947), 1011–1014; **247** (1958), 1307–1310 [MR **18**, 857; **21** #1206].
J. Moser (Cambridge, Mass.)

6266:
Sconzo, Pasquale. Il calcolo della posizione e della velocità nel problema della determinazione di un'orbita. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 422–427.

In this paper we find the derivation of simple formulae to obtain the position and velocity for the middle observation of a body moving about the sun under Newtonian attraction in the problem of orbit determination, given three observations of α and δ. These are obtained by direct substitutions of the well-known f and g series in the Laplacian method into expressions for the topocentric coordinates, thus eliminating geocentric parallax.

The coordinates and velocities are obtained by successive approximation. A preliminary orbit for a body with small eccentricity in which the range of observation does not exceed 30 days requires up to the fifth power of $\tau=k(t-t_0)$. The main advantages of this method which has been used extensively are: (1) simple formulae suited to digital computers (2) no necessity for special tables, (3) the trigonometric functions of α and δ are calculated only once. *M. S. Davis* (New Haven, Conn.)

6267:

Battin, Richard H. The determination of round-trip planetary reconnaissance trajectories. J. Aero/Space Sci. **26** (1959), 545–567.

"Some interesting geometrical and analytical properties of free fall trajectories are developed and subsequently exploited to provide a technique for determining interplanetary transfer orbits. After a general discussion, the determination of propulsion-free round trips to the planet Mars is considered. As a preliminary step, the reconnaissance problem is analyzed for a simple two-dimensional model of the solar system, assuming circular planetary orbits.

"Finally, a method is described for determining round-trip, non-stop reconnaissance trajectories in a three-dimensional model with elliptical planetary orbits. The results from the simplified approach are compared with those obtained from the true model. It is found that several important features of the trajectory problem are basically three-dimensional in nature and that the simplified model is inadequate for their description." (Author's summary.) *P. O. Bell* (Culver City, Calif.)

6268:

Upton, E.; Bailie, A.; and Musen, P. Lunar and solar perturbations on satellite orbits. Science **130** (1959), 1710–1711.

6269:

★Spitz, Armand; and Gaynor, Frank. Dictionary of astronomy and astronautics. Philosophical Library, New York, 1959. vii + 439 pp. (16 plates) $6.00.

A non-mathematical dictionary, useful to the general reader.

6270:

★Walker, A. G. Axioms for cosmology. The axiomatic method. With special reference to geometry and physics. Proceedings of an International Symposium held at the Univ. of Calif., Berkeley, Dec. 26, 1957–Jan. 4, 1958 (edited by L. Henkin, P. Suppes and A. Tarski), pp. 308–321. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 1959. xi + 488 pp. $12.00.

A system of axioms for cosmology is presented from which the following two theorems can be deduced: (i) the events in the history of a fundamental particle are "linearly" ordered, i.e., can be parametrized by a single real parameter; (ii) the set of fundamental particles can be given the topology and structure of a geodesic metric space [H. Busemann, *The geometry of geodesics*, Academic Press, New York, 1955; MR 17, 779] in such a way that the metric has the properties required in cosmology.

These axioms lead to Robertson-Walker spaces whose 3-curvature $k \leq 0$. Spaces with $k>0$ need a more complicated system of axioms. *D. W. Sciama* (London)

6271:

Kašanin, R. Réfraction astronomique moyenne. Glas Srpske Akad. Nauka **232**. Od. Prirod.-Mat. Nauka (N.S.) **15** (1958), 1–8. (Serbo-Croatian. French summary)

GEOPHYSICS

See also 6060, 6143, 6158.

6272:

Scheidegger, Adrian E. Mathematical methods in geology. Amer. J. Sci. **258** (1960), 218–221.

Defense of quantitative methods, as in the author's *Principles of Geodynamics* [Springer, Berlin, 1958; MR **20** #759].

6273:

Gates, W. Lawrence. On the truncation error, stability, and convergence of difference solutions of the barotropic vorticity equation. J. Meteorol. **16** (1959), 556–568.

For an initial value problem represented by a linear differential equation with constant coefficients in a single dependent variable and a harmonic initial condition, the author examines the conditions under which the truncation error due to replacing the differential equation with a difference equation approaches zero as the time and space increments approach zero. For the one dimensional barotropic vorticity equation which is a simple dynamic model of the atmosphere, an explicit analytic solution is obtained and compared with various finite difference solutions. In addition, stability and convergence properties of the finite difference formulations are examined.

The time differencing schemes considered with centered space differences include the forward difference, the first-forward-then-centered difference and the implicit difference cases. The first of these proves unstable, the second conditionally stable and the third stable. In each difference scheme considered the truncation error leads to a cumulative phase departure of the difference solution relative to the true solution, an effect which is approximately proportional to the cube of the wavelength. In the implicit difference case there is no restriction on the permissible values of the time increment for a selected space increment except that dictated by the tolerable truncation effects. Consequently, practical usefulness of this method in meteorological dynamics will depend on its design with small truncation error. *J. Blackburn* (New York, N.Y.)

6274:

Dorman, James; Ewing, Maurice; and Oliver, Jack. Study of shear-velocity distribution in the upper mantle by mantle Rayleigh waves. Bull. Seismol. Soc. America **50** (1960), 87–115.

6275:

Molodensky, M. S. Deducing Stokes's formula with a relative error of the order of the square of the flattening. Bull. Géodésique (N.S.) no. 50 (1958), 50–51.

The classical formula of G. G. Stokes for the height T of the geoid above the ellipsoid in terms of the gravity anomalies Δg was originally derived for a sphere. Molodensky calculated the corrections which take account of the first power of the flattening, f, i.e., of the square of the eccentricity e of the meridian section.

The solution is obtained from a linear integral equation

$$2\pi T = a\int \frac{\Delta \bar{g}}{2\sin\psi_2}\,d\omega^\sigma + \int \bar{T}(f_0+e^2f_1+e^4f_2+\cdots)\,d\omega^\sigma$$

in which $\bar{T}$ and $\Delta\bar{g}$ represent T, Δg, modified by terms of order e^2; $d\omega^\sigma$ is a surface element, ψ the angular distance from the point at which $\bar{T}$ is to be found to an arbitrary point of the surface; a the earth's semi-major axis; and the f's are known functions of latitude and ψ.

The equation is solved as a series in powers of e^2; results only are given, and only for the zero-power (Stokes solution) and first-power terms. The first-power term is a complicated function of the Stokes function. The result is stated to have been checked by numerical methods. There are numerous misprints.

J. A. O'Keefe (Chevy Chase, Md.)

6276:

Hazay, I. Ausführung der statischen Koordinatenausgleichung. I. Ausgleichung eines Punktes. Acta Tech. Acad. Sci. Hungar. **23** (1959), 397–430.

Let v_i be the correction to the measured length or direction of a line in a triangulation net. If v_i is normalized, it can be interpreted as a force, R_i, attached to each line. Setting $\sum R_i=0$ imposes a so-called static (equilibrium) condition, which directly assigns to each line a weight proportional to the measured length or direction, and provides the basis for the adjustment of the net.

The author recapitulates his earlier papers on static adjustments [Mitt. Berg. Hütt. Abt. Ung. Univ. Tech. Wirtschaftwiss. Sopron **11** (1939); **14** (1942)], and applies the method to the adjustment of individual points by intersection or resection, with either (or both) measured lengths or directions. The generality of the method is indicated by its ability to handle specially weighted measurements, and special condition equations. A number of examples are worked out in detail.

B. Chovitz (Washington, D.C.)

6277:

Hazay, I. Ausführung der statischen Koordinatenausgleichung. II. Gemeinsame Ausgleichung mehrerer Punkte. Acta Tech. Acad. Sci. Hungar. **23** (1959), 431–472. (English, French and Russian summaries)

A continuation of the preceding paper, extending the method to the simultaneous adjustment of several points. The bulk of the article is devoted to the detailed solution of several examples, the most complicated one involving 9 points and 27 measured lengths.

B. Chovitz (Washington, D.C.)

OPERATIONS RESEARCH, ECONOMETRICS, GAMES

See also 6101, 6269

6278:

Gorman, W. M. Separable utility and aggregation. Econometrica **27** (1959), 469–481.

Utility theory explains the consumption vector x, of which the components are nonnegative rates of consumption of various goods, as the vector which maximizes a strictly concave utility function $u(x)$ subject to $p\cdot x=Y$, where p is a vector of market prices, and Y the money income of the choosing individual. $u(x)$ is called "separable" if x can be partitioned into subvectors x_r such that $u(x)=F(v_1(x_1),\cdots,v_n(x_n))$, where the v_r are scalar functions. Writing p_r for corresponding subvectors of p, the paper studies conditions, on a separable $u(x)$, for (A) "perfect aggregation" and for (B) "local aggregation" of prices. (A) is defined as the existence of scalar functions (price indices) $P_r(p_r)$ such that expenditure on the rth commodity group depends on Y and the P_s only,

$$(3)\qquad y_r = p_r\cdot x_r = \phi_r(P_1, P_2, \cdots, P_n, Y).$$

(B) is defined in differential form by

$$(6)\qquad dy_r = \sum_s \sum_i a_{rs}(p, Y)w_{si}(p, Y)dp_{si}+a_{r0}(p, Y)dY,$$

where p_{si} is the ith component of p_s. In case (A) the consumer can make budget allocations y_r for all commodity groups on the basis of the P_s and Y alone. In case (B) he can approximate the changes Δy_r from previous utility-maximizing allocations, given ΔY and differential price indices $\sum_i w_{si}\Delta p_{si}$.

The following criteria are shown to hold provided the maximizing x has only positive components. (B) occurs if and only if F has one of the three forms $F^*=F(v_1, v_2)$, $F^{**}=F(v_1,\ f(v_1,\ \cdots,\ v_n))$, $F^{***}=v_1+\cdots+v_d+f(v_{d+1},\cdots,v_n)$, $0\leq d<n$, and every v_r appearing as an argument of an f homogeneous of degree one in x_r.

Conditions sufficient for (A), and also necessary except when either $n=2$ or exactly one of the $v_r(x_r)$ is not homogeneous, are: (33) F has the form F^{***} above, (34) $v_m(x_m)$ is homogeneous of degree one for $m=d+1,\cdots,n$, and (35) for all s the maximum of $v_s(x_s)$ subject to $p_s\cdot x_s=y_s$ satisfies $v_s=G_s(y_s/P_s(p_s))+Q_s(p_s)$, where P_s and Q_s are homogeneous of degrees one and zero, respectively.

The implications of these conditions in terms of consumers' behavior are discussed.

{A few remarks to assist the unwary reader: (2) contains misprints. From (15) and (16) the components in the sth position should be omitted. (39) is a statement to be proved in section II,2.}

T. C. Koopmans (New Haven, Conn.)

6279:

Rashevsky, N. Some remarks on the mathematical aspects of automobile driving. Bull. Math. Biophys. **21** (1959), 299–308.

Some simple relations are derived from consideration of the fact that the steering of a car is subject to fluctuations,

that roadways have finite width, and that a driver has a minimum reaction time which serves as a lower bound on the time required to make corrections for errors in steering. If the velocity of a car is too high, a fluctuation in lateral position will exceed the width of the road before corrections can be made. Some simple models for passing are also considered. *G. Newell* (Providence, R.I.)

6280:

Rufener, Ernst. Kleine Bemerkung zu einer Funktionalgleichung der Prämienreserve. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 21–28. (French, Italian and English summaries)

If $V(t, x, n)$ denotes the reserve at time t on an n-year continuous premium endowment insurance of unity issued at age x, it is shown that

$$\frac{\partial V}{\partial t}-\frac{\partial V}{\partial x}+\frac{\partial V}{\partial n}=\left[\frac{\partial V}{\partial t}\right]_{t=0}\frac{\bar{a}_{x+t:\overline{n-t}|}}{\bar{a}_{x:\overline{n}|}}.$$

{In the paper a minus sign is erroneously placed before the right-hand member of this equation.} This relation is used to show that the hyperbolic arc with one vertical asymptote used by Jecklin and Strickler [Mitt. Verein. Schweiz. Versich.-Math. **54** (1954), 71–80] and Strickler [ibid. **55** (1955), 83–99] to approximate the reserve would actually yield the exact values on the basis of a possible mortality law. *T. N. E. Greville* (Kensington, Md.)

6281:

Bühlmann, Hans. Die beste erwartungstreue lineare Schätzfunktion der Übersterblichkeit. Mitt. Verein. Schweiz. Versich.-Math. **59** (1959), 49–58. (French, Italian and English summaries)

A special case of substandard mortality is considered where the extra mortality for all ages is a constant percentage of the normal mortality.

In the class of all linear unbiased estimates of that percentage of extra mortality the most efficient estimate depends on the quality to be estimated. The minimax solution is preferred as the best possible estimate, minimizing the upper limit of the variance of the estimate.

P. Johansen (Copenhagen)

6282:

Pistoia, A. Dirichlet transforms applied to interest functions. Skand. Aktuarietidskr. **1958**, 167–171 (1959).

Der Verfasser löst mit Hilfe der Dirichlet'schen Transformierten die Differenzengleichung

$$\sum_{n=0}^{p}\alpha_n y[t+p-n]=A(t).$$

Die Koeffizienten $\alpha_1, \cdots, \alpha_p$ sind Funktionen des Zinsfusses und unabhängig von t, $\alpha_0\neq 0$. *W. Saxer* (Zürich)

6283:

Philipson, Carl. A contribution to the problem of estimation involved in an insurance against loss of profit. Skand. Aktuarietidskr. **1958**, 71–92 (1959).

Der Verfasser betrachtet Versicherungen zur Deckung von Ausfällen im Einkommen (Beispiel: Ergebnis einer Ernte). Wenn das Einkommen unter eine bestimmte Grenze K fällt, entsteht ein Schaden. Dieser Schaden wird als die Differenz eines Betrages Q und des mittleren Einkommens in der Versicherungsperiode definiert. Q stellt die Differenz zwischen dem mittleren Einkommen und dem Selbstbehalt dar. $Q-K$ ist die Höhe der Franchise, wenn dieser Betrag positiv ist.

Der Verfasser zeigt, wie man auf Grund der Theorie der Zeitreihen und der Methoden der Demand-Analysis für eine solche Versicherung schätzen kann.

W. Saxer (Zürich)

6284:

Shetty, C. M. A solution to the transportation problem with nonlinear costs. Operations Res. **7** (1959), 571–580.

In the transportation problem discussed in this paper, the cost of producing the considered good at a typical source is a given nonlinear function of the amount to be produced, while the shipping cost for the typical route is strictly proportional to the amount shipped over this route. Production and shipping programs are to be determined that minimize the total cost while satisfying known demands. Optimality conditions that could have been obtained from the general theory of non-linear programming [see, for instance, A. W. Tucker, Operations Res. **5** (1957), 244–257; MR **19**, 618] are derived from first principles. An iterative method of solution is presented that proceeds from a feasible solution to one of lower cost. Several variants of the problem are discussed.

W. Prager (Providence, R.I.)

6285:

Sinden, Frank W. Mechanisms for linear programs. Operations Res. **7** (1959), 728–739.

The paper describes mechanical models for linear programming problems with coefficient matrices that consist of plus ones, minus ones, and zeros. While unavoidable imperfections may prevent proper functioning, these models are nevertheless useful because they afford an intuitive insight into principles and methods of linear programming. Thus, the primal-dual method of Dantzig, Ford, and Fulkerson [*Linear inequalities and related systems*, pp. 171–181, Princeton Univ. Press, 1956; MR **19**, 719] has a particularly simple interpretation in terms of one of these models. Other models are given for the transportation problem. (The models may also serve as a reminder that extremum problems with one-sided constraints have been successfully treated in analytical mechanics long before the modern development of linear programming.) *W. Prager* (Providence, R.I.)

6286:

Lambert, François. Programmes linéaires: méthodes et exemples. Cahiers Centre Math. Statist. Appl. Sci. Social. **1**, 7–70 (1959).

The paper is written as an introduction to linear programming. After presenting typical applications (transportation, assignment, diet), the author discusses important concepts and methods relying largely on numerical examples that involve only a few variables.

W. Prager (Providence, R.I.)

6287:

Galliher, H. P.; Morse, Philip M.; and Simond, M. Dynamics of two classes of continuous-review inventory systems. Operations Res. **7** (1959), 362–384.

Analysis of the policy of ordering a fixed quantity whenever stock on hand and on order falls below a predetermined level, mainly for Poisson demand and exponential delivery time distributions.

M. J. Beckmann (Providence, R.I.)

6288:

Ash, Milton. Diffusion attrition model. Operations Res. 8 (1960), 82–89.

The author indicates how a mathematical model of combat between attacking and defending forces can be formulated in classical diffusion terms.

R. Bellman (Santa Monica, Calif.)

6289:

Schwartz, Benjamin L. Solution of a set of games. Amer. Math. Monthly **66** (1959), 693–701.

The paper describes and solves a family of zero-sum two-person games which are sufficiently complicated to require a game-theoretic analysis yet sufficiently simple so that the analysis is illuminating. These games should be helpful to both teachers and research workers in need of non-trivial examples. *K. O. May* (Northfield, Minn.)

6290:

★Nering, Evar D. Symmetric solutions for general-sum symmetric 4-person games. Contributions to the theory of games, Vol. IV, pp. 111–123. Annals of Mathematics Studies, no. 40. Princeton University Press, Princeton, N.J., 1959. xi+453 pp. $6.00.

Symmetric solutions are given for all general-sum symmetric four-person games. Some of these specialize, in the zero-sum case, to solutions given by von Neumann and Morgenstern. *J. H. Blau* (Stanford, Calif.)

6291:

Newman, Donald J. A model for 'real' poker. Operations Res. 7 (1959), 557–560.

The model differs from the last one treated by J. von Neumann and O. Morgenstern [*Theory of games and economic behavior*, 2nd ed., Princeton Univ. Press, 1947; MR **9**, 50; pp. 215–218] in allowing the first player to bid any amount in $[1, \infty)$. The solution is similar; the value goes up from 1/9 (when the bid must be 1 or 3) to 1/7.

J. Isbell (Lafayette, Ind.)

6292:

Huyberechts, Simone. La théorie des jeux. Cahiers Centre Math. Statist. Appl. Sci. Social. **1**, 71–90 (1959).

An expository article presenting minimax theory for games with monetary payments. There is no mention of utility theory. *J. Isbell* (Lafayette, Ind.)

BIOLOGY AND SOCIOLOGY

See also 6054.

6293:

Robertson, Alan. The sampling variance of the genetic correlation coefficient. Biometrics **15** (1959), 469–485.

Two characters (X, Y) are observed for each individual of N genetic groups of n members each. The genetic correlation coefficient, r_g, is that between the true group means. In the special case when the intraclass correlations are the same for X and Y, r_g may be estimated by

$$\hat{r}_g = (A-C-B+D)/(A-C+B-D),$$

where A, B, C, D are the group, group × character, individuals within groups and remainder mean squares respectively. When X and Y are measured on different individuals, $\hat{r}_g = (A-B)/(A+B-2C)$. The variances of the $\hat{r}_g$'s are calculated for the case of normal distributions and large N. Special cases and the genetic implications of the formulae are considered in detail.

M. Stone (Cambridge, England)

6294:

Mode, C. J.; and Robinson, H. F. Pleiotropism and the genetic variance and covariance. Biometrics **15** (1959), 518–537.

To deal with situations in which the same gene may affect unrelated characters, the concept of genetic variance is extended to genetic covariance, and it is shown that the latter can be partitioned in exactly the same way as the former into additive, dominant and epistatic components. Genetic, genotypic and phenotypic coefficients of correlation are then defined, and the calculation and interpretation of these and other parameters are illustrated. The arguments and procedure generalize those of Comstock and Robinson [Biometrics **4** (1948), 254–265].

I. M. H. Etherington (Edinburgh)

6295:

Tallis, G. M. Sampling errors of genetic correlation coefficients calculated from analyses of variance and covariance. Austral. J. Statist. **1** (1959), 35–43.

Genetic correlations can be estimated by taking family groups and comparing the components of variance and covariance within and between groups. In this paper the groups considered are sets of k half-sibs, where k is supposed constant. An expression is given for the standard error of the estimated correlation. If k varies, this expression underestimates the standard error. Tables are given of the values of k which minimize the standard error, for various values of the phenotypic and genotypic correlations. These tables can also be used with groups of full sibs. *C. A. B. Smith* (London)

6296:

Neyman, Jerzy; and Scott, Elizabeth L. Stochastic models of population dynamics. Science **130** (1959), 303–308.

An interesting discussion of the use of stochastic models illustrated in terms of the authors' work on models for the struggle for existence between two species of flour beetles [Neyman, Park and Scott, Proc. 3rd Berkeley Sympos. Math. Statist. Probability, 1954–1955, vol. IV, pp. 41–79, Univ. California Press, Berkeley-Los Angeles, 1956; MR **18**, 951] and models applicable to cosmology [Neyman and Scott, J. Roy. Statist. Soc. Ser. B **20** (1958), 1–43; MR **21** #4051]. The latter model is a stochastic process which applies to a variety of clustering phenomena. Its applications to biological population, bombing, cloud chambers, and cosmology are discussed. In the discussion of cosmology, the authors suggest that many

of the difficulties that occur in the theoretical study of the universe arise from the deterministic nature of the theories and that these difficulties disappear when a stochastic model for the universe is considered.

J. L. Snell (Hanover, N.H.)

INFORMATION AND COMMUNICATION THEORY

See also 6026.

6297:

Kolmogorov, A. N. Theory of information transmission. Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat-Fiz. (3) **13** (1959), no. 1 (28), 5–33. (Romanian)

Romanian translation of Kolmogorov's work published in the collection *Sessiya Akademii Nauk SSSR po naučnym problemam avtomatizacii proizvodstva, 15–20 Oct. 1956, Plenarnye zasedaniya*, pp. 66–99 [Izdat. Akad. Nauk SSSR, Moscow, 1957].

Chapter I deals with the origin and object of the theory and begins with some considerations on the estimation of the amount of information and certain tabulation problems concerning continuous functions. The notions of transmission capacity and rate of creating information are then discussed for noiseless channels. Further, the entropy and the amount of information are introduced for the discrete case. The transmission capacity is then considered for a noisy channel and a brief discussion on Shannon's fundamental concepts is given.

Chapter II is concerned with the foundations of the theory for continuous communications. Firstly, the amount of information $I(\xi, \eta)$ contained in one random object ξ relative to the random object η is considered and an explicit formula for two Gaussian random vectors is given. After sketching the foundations of Shannon's theory, the ε-entropy is evaluated for certain particular cases and then the amount of information and rate of creating information is discussed in connection with stationary processes. Certain evaluations are given by means of spectral theory. Finally, some evaluations of the transmission capacity for certain special cases are given.

Chapter III contains a brief discussion on the limits within which the notion of amount of information may be applied.

The work has mainly an expository character.

[A Hungarian version of the same work appears in Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 113–142; MR **20** #4450. A German version, without chapter III, is inserted in *Arbeiten zur Informationstheorie I*, pp. 91–116, VEB Deutscher Verlag der Wissenschaften, Berlin, 1957. For chapter II, see also A. N. Kolmogorov, Trans. I.R.E. **IT-2** (1956), 102–108.]

R. Theodorescu (Bucharest)

6298:

Dobrušin, R. L. A general formulation of Shannon's fundamental theorem in the theory of information. Dokl. Akad. Nauk SSSR **126** (1959), 474–477. (Russian)

Statement of theorems proved in #6299 below.

J. Wolfowitz (Ithaca, N.Y.)

6299:

Dobrušin, R. L. A general formulation of the fundamental theorem of Shannon in the theory of information. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 3–104. (Russian)

This paper gives the proofs of several coding theorems designed to realize the program in Kolmogoroff's formulation of the Shannon theory of transmission of information. This formulation is given in Kolmogoroff's paper #6297 above. It is impractical to state or describe any of these theorems within a reasonable compass. The author states: "... the verification of our conditions for specific classes of processes appears to be a non-trivial problem not yet solved in principle."

Kolmogoroff's formulation is given in Section 2 of his paper cited above. In this formulation the input message is a chance variable with a specified distribution. In the work of Shannon and other American writers, the choice of input message is completely arbitrary. [This is clearly to be seen in, for example, C. E. Shannon, Bell System Tech. J. **38** (1959), 611–656; MR **21** #1920; or A. Feinstein, *Foundations of Information Theory*, McGraw-Hill, New York, 1958; MR **20** #1594; p. 43; or J. Wolfowitz, Illinois J. Math. **3** (1959), 477–489.]

J. Wolfowitz (Ithaca, N.Y.)

6300:

Rényi, A. On the dimension and entropy of probability distributions. Acta Math. Acad. Sci. Hungar. **10** (1959), 193–215. (Russian summary, unbound insert)

Let the entropy of a discrete random variable be denoted by $\mathbf{H}_0(\eta)=\sum_1^\infty q_k \log 1/q_k$, where $q_k=\mathbf{P}(\eta=y_k)$, $k=1, 2, \cdots$. For any real-valued random variable ξ put $\xi_n=[n\xi]/n$, where $[x]$ denotes the integer part of x. The upper and lower dimensions of (the distribution of) ξ are defined as $\bar{\mathbf{d}}(\xi)=\limsup_n \mathbf{H}_0(\xi_n)/\log n$ and $\underline{\mathbf{d}}(\xi)=\liminf_n \mathbf{H}_0(\xi_n)/\log n$. If $\mathbf{H}_0([\xi])<\infty$, then $0\leqq \underline{d}(\xi)\leqq \bar{d}(\xi)\leqq 1$; the author shows that all inequalities are possible. When $\underline{\mathbf{d}}(\xi)=\bar{\mathbf{d}}(\xi)$ the common value is defined as the dimension of ξ. When ξ has a definite dimension, say $\mathbf{d}(\xi)=d$, the d-dimensional entropy of ξ is defined as $\mathbf{H}_d(\xi)=\lim_n(\mathbf{H}_0(\xi_n)-d\log n)$, provided the limit exists. For discrete random variables, if $\mathbf{H}_0([\xi])<\infty$ then $\mathbf{d}(\xi)=0$ and $\mathbf{H}_d(\xi)$ for $d=0$ coincides with $\mathbf{H}_0(\xi)$ as originally defined. If the distribution of ξ is absolutely continuous with density $f(x)$, then $d(\xi)=1$ and $\mathbf{H}_1(\xi)=\int f(x)\log 1/f(x)dx$. Suppose the distribution function of ξ has the form $F(x)=(1-d)F_0(x)+dF_1(x)$, where F_0 is purely discrete with probabilities p_k, F_1 is absolutely continuous with density $f_1(x)$, and $0<d<1$. Then ξ has dimension $d(\xi)=d$, and $\mathbf{H}_d(\xi)=(1-d)\sum p_k\log 1/p_k+d\int f_1(x)\log 1/f_1(x)\cdot dx+d\log(1/d)+(1-d)\log(1/(1-d))$. The author remarks that the question about the existence of dimension and entropy in the general (one-dimensional) case seems to be rather intricate. The author describes generalizations for vector-valued random variables and for random variables taking values in a compact metric space, relating the latter to the ε-entropy notion of Kolmogorov [#6297 above].

S. P. Lloyd (Murray Hill, N.J.)

6301:

Zimmerman, Seth. An optimal search procedure. Amer. Math. Monthly **66** (1959), 690–693.

A finite set of objects with associated probabilities is

given. It is required to determine the identity of a randomly selected object using a minimum expected number of yes-or-no questions. This problem was first solved (in an obviously equivalent form) by D. A. Huffman [Proc. I.R.E. **40** (1952), 1098–1101].

S. W. Golomb (Pasadena, Calif.)

6302:

Brillouin, L. Information theory and the divergent sums in physics. Ann. Physics **5** (1958), 243–250.

"The author has previously shown, by the methods of information theory, that there is no actual limit to the small distances that can be measured, but that the cost of the measurement increases enormously when distances become really small. This strongly suggests the introduction of a probability factor in sums containing small distances. Such an attempt is made, and yields a type of summation closely related to the Cesaro sums. The method leads to the definition of a characteristic length $h/2m_0c$ for a particle of mass m_0. For an electron this is $\frac{1}{2}$ the Compton length, and the electromagnetic mass is of the order of 1/860 of the total mass . . . " (From the author's summary)

R. A. Leibler (Princeton, N.J.)

6303:

Braffort, Paul; et Castagne, René. La fréquence instantanée complexe. Définition et mesure. C. R. Acad. Sci. Paris **249** (1959), 854–856.

This paper is concerned with defining a frequency for a signal $s(t)$ even if $s(t)$ is not repetitious. To do this, a function $\sigma(t)$ orthogonal to $s(t)$ is invoked to form $S(t)=s(t)+i\sigma(t)$, and then the "instantaneous complex frequency" $\sigma(t)+i\omega(t)$ is defined to be the logarithmic derivative of $S(t)$. In the case of a periodic function this reduces to $2\pi i$ times the frequency.

Means of measuring the instantaneous complex frequency are discussed, and a circuit which will give the two components is diagrammed.

H. H. Campaigne (Jessup, Md.)

6304:

Andreev, N. I. General condition of extremum of given function of mean-square error and of mathematical expectation square of error of dynamic system. Avtomat. i Telemeh. **20** (1959), 833–838. (Russian. English summary)

Given stochastic processes $X(t)$, $Y(t)$, and a goodness-of-fit function $f(p, q)$, it is desired to find an operator A, sending $X(t)$ into a function of s, such that $f(p, q)$ assumes an extremum value. Here $p=\{M[Y(s)-AX(t)]\}^2$, $q=M\{[Y(s)-AX(t)]^2\}$, where M denotes expectation. The hypothesis is made that the function f has the property that, for fixed $q=c$, f assumes its extremal value when A is chosen so that $p+\lambda q$ is minimized. λ is a parameter whose optimum value is left open until later. The problem of minimizing $p+\lambda q$ is solved in the case when the set of random variables $\{AX(t)\}$, at s, forms a linear manifold as A ranges over the set of permissible operators, and leads in a case of general interest to an integral equation. (A need not be a linear operator.) For the case $\lambda=\infty$, which occurs if f depends only on q, the results reduce to those of previous workers.

E. Reich (Minneapolis, Minn.)

6305:

Good, I. J.; and Doog, K. Caj. A paradox concerning rate of information: corrections and additions. Information and Control **2** (1959), 195–197.

Supplement to the authors' paper in Information and Control **1** (1958), 113–126 [MR **19**, 1245].

E. Reich (Minneapolis, Minn.)

SERVOMECHANISMS AND CONTROL

See also 5784, 5791, 6198, 6199.

6306:

★Цыпкин, Я. З. Теория импульсных систем. [Cypkin, Ya. Z. Theory of impulse systems.] Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958. 724 pp. 23.25 rubles.

This is a definitive treatise on linear sampled-data systems by a leading Soviet control theorist. It covers exhaustively both the basic theory of discrete-time systems and a variety of specialized techniques centering on the analysis and design of sampled-data servomechanisms. Of particular interest are the treatments of the following topics. In chapter 3, the analysis of non-feedback systems containing two or more samplers operating at different frequencies with arbitrary relative phases; systems with time-varying parameters, including pulse-width modulated controllers; systems subjected to random inputs; prediction and filtering. In chapter 5, the stability of feedback systems; analysis of feedback systems containing two or more non-synchronous samplers; prediction and filtering of processes with unknown expectation; oscillations in nonlinear sampled-data systems; minimal-time systems. Numerous applications of the techniques developed in chapters 3 and 5 are discussed in detail in chapters 4 and 6. The appendix contains an extensive table of z-transforms and tables of trigonometric and exponential functions.

Although the text provides a most complete and up-to-date treatment of the theory of sampled-data systems, there are a few important topics which are not included within its compass. Among these are the casting of input-output relationships into a form which places in evidence the state of the system, problems of optimal control (using dynamic programming and related techniques), and the analysis of systems with periodically varying parameters. These omissions do not detract appreciably from the great value of this text as a reference for the designer of sampled-data systems and the advanced student of control theory.

L. A. Zadeh (Berkeley, Calif.)

6307:

Woodrow, R. A. On finding a best linear approximation to system dynamics from short duration samples of operating data. J. Electronics Control (1) **7** (1959), 176–192.

6308:

Matveev, P. S. Synthesis of servosystem compensation devices with noise. Avtomat. i Telemeh. **20** (1959), 721–728. (Russian. English summary)

In a previous paper [Avtomat. i Telemeh. **16** (1955), 233–257] V. V. Solodovnikov and the author determined

the solution for the optimum finite memory compensation for a feedback system subject to stationary stochastic inputs with two points of application. The performance criterion was minimum mean square error. The present paper is an extension of the earlier one to the case in which deterministic signals are also applied to one of the two points at which the stochastic signals enter the system. This paper is closely related to the extension of Wiener's work developed by Zadeh and Ragazzini [J. Appl. Phys. **27** (1950), 645–655; MR **12**, 347].

P. M. De Russo (Troy, N.Y.)

6309:

Moisil, Gr. Equations caracteristiques d'un relais. Acad. R. P. Romîne. Fil. Cluj. Stud. Cerc. Şti. Ser. I. **6** (1955), no. 3–4, 7–15. (Romanian. Russian and French summaries)

Let $\xi^{(i)}$ be a symbol which is 1 if the ith winding of a given relay transmits the current, 0 otherwise. Let $x^{(i)}$ be a symbol which is 1 if the ith contact of the relay is closed, 0 if it is open. Combine 0 and 1 as elements of a two-element Boolean algebra. Let $x_N^{(i)}$ be $x^{(i)}$ at time N (likewise for $\xi_N^{(i)}$). Under certain circumstances, x_N can be expressed in terms of earlier x's and ξ's. Such expressions (or characteristic equations) are worked out for about 10 different types of relays. A general reference is to the author's paper, Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 7–53 [MR **17**, 328].

F. Haimo (St. Louis, Mo.)

6310:

Nemitz, William; and Reeves, Roy. A mathematical theory of switching circuits. Math. Mag. **33** (1959/60), 1–6.

Expository.

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